

# A class of asymmetric gapped Hamiltonians on quantum spin chains and its characterization I

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## Abstract

We introduce a class of gapped Hamiltonians on quantum spin chains, which allows asymmetric edge ground states. This class is an asymmetric generalization of the class of Hamiltonians in [FNW]. It can be characterized by five qualitative physical properties of ground state structures. In this Part I, we introduce the models and investigate their properties.

## 1 Introduction

Recently, gapped ground state phases attract a lot of attention. It is related to various fields of mathematics and physics, including condensed matter physics, quantum information, spectral theory, and topology, and studied widely from many different points of view. Even if we restrict our attention to quantum spin systems, many interesting facts have been found in the last decades. One of the most famous discoveries is the area law in one dimensional quantum spin system proven by Hastings [H]. In the paper, he showed and used the fact that a unique ground state of a gapped Hamiltonian can be approximated by a product of three localized operators. This fact holds even in more general setting, see [HMNS]. Furthermore, the exponential decay of correlations of gapped ground states was proven in [HK], [NS]. In a word, the existence of the gap guarantees us to have a good control on the spectral projection of ground state spaces.

Such a nice control was used to show the automorphic equivalence of the ground state structures in [BMNS], in the classification problem of gapped Hamiltonians. Here, two Hamiltonians are defined to be in the same class, if they are the endpoints of a  $C^1$ -path of Hamiltonians along which the spectral gap above the ground state energy does not close. (See [BO] for a more formal definition.) We would like to emphasize that in [BMNS], finite volume Hamiltonians with open boundary conditions are considered. Therefore, we call this classification, the  $C^1$ -classification with open boundary condition. One benefit of considering open boundary condition is that it possesses the information of edge states, as well as the bulk one. Another more technical advantage is that it is convenient when we use the martingale method introduced in [N] to show the spectral gap.

As we have seen, one can derive strong results under quite general setting, *if we assume the existence of the spectral gap*, because of the nice control of the spectral projections. However, *to prove the existence of the gap itself* turns out to be a much more difficult problem, especially in more than one dimensional systems. For one dimensional systems, a recipe to construct Hamiltonians out of  $n$ -tuple of matrices in some auxiliary systems is known. (See Subsection 1.2.) We would like to call the Hamiltonians given by this recipe, the MPS (matrix product state) Hamiltonians. Just a random choice of the  $n$ -tuples does not guarantee the spectral gap. In [FNW], a sufficient condition,

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which we call the injectivity condition, to guarantee the gap was introduced. (See Remark 1.15.) In this setting of [FNW], the bulk ground state turns out to be a pure finitely correlated state (or MPS). The class of Hamiltonians given in this way with the injectivity condition covers the AKLT model [AKLT]. Classification of this class of Hamiltonians was studied in [CGW1], [CGW2], [SPC], [BO].

Having this nice recipe, one natural and basic question is how much of the gapped ground state phase in quantum spin chains is covered by the Hamiltonians given by MPS recipe with the injectivity condition. More precisely, for each equivalence class of the  $C^1$ -classification, is there a representative given by such a Hamiltonian?

The answer is no: If we assume the injectivity condition, the edge states (namely, the ground state space on left/right half infinite chains) have to be symmetric [BO]. However, in [BN1, BN2], a particular family of gapped models called PVBS models, with asymmetric edge states was introduced. PVBS model still can be given by the MPS recipe, but the matrices do not satisfy the injectivity condition. In this model, the bulk ground state is a pure product state. It is then natural to explore a general condition on  $n$ -tuple of matrices which guarantees the spectral gap, but still allows the asymmetric edge ground states.

This is what we do in this series of papers. In Part I, we introduce a new class of MPS gapped Hamiltonians which covers Hamiltonians in [FNW]. This class allows asymmetric edge ground states. We investigate the properties of this Hamiltonian in this paper (Theorem 1.18). This new class is not a mere generalization of the known models. In Part II, we will show that this class has a characterization in terms of five qualitative physical conditions (which corresponds to (i)-(v) and (viii)) of Theorem 1.18). More precisely, we will show if a (not necessarily MPS) Hamiltonian satisfies these five conditions, there is an MPS Hamiltonian from our class satisfying the followings: The ground state spaces of the two Hamiltonians on the infinite intervals coincide. In the finite intervals, the spectral projections onto the ground state space of the original Hamiltonian on each intervals are well approximated by that of the MPS Hamiltonian. This last property has a corollary to the  $C^1$ -classification problem i.e., the classification problem of Hamiltonians satisfying these five properties is reduced to the classification problem of our generalized class of MPS Hamiltonians. The benefit is that the latter one has a concrete structure which allows us to handle the spectral gap.

We should emphasis two points. Firstly, it is well known that any vector state on a finite interval can be represented as an MPS. However, naive representation would require the size of the auxiliary systems to grow very fast, as the length of the intervals goes to infinity. If it is the unique ground state of a gapped Hamiltonian or even just by having the exponential decay of the correlation functions, the growth can be reduced significantly [H], [BH]. What we do in this paper is however, of different nature. We would like to have the dimension of the auxiliary system to be fixed. As a cost, we have to assume that our Hamiltonian is frustration free. The second point is that if we care only about the ground state in the bulk, it is already known that any ground state of frustration free Hamiltonian with uniformly bounded degeneracy is an MPS [M1], [M2]. The difference here is that we would like to care about the spectral gap, the  $C^1$ -classification, and the edge states. As a result, we have to represent not only bulk ground state, but also left/right edge ground states and the ground states on the finite intervals, simultaneously, using the *same* auxiliary system. This requires the detailed analysis of the auxiliary system.

We will use the notations listed in Appendix A freely.

## 1.1 Hamiltonians and ground state structures

For  $\mathbb{N} \ni n \geq 2$ , let  $\mathcal{A}$  be the finite dimensional  $C^*$ -algebra  $\mathcal{A} = M_n$ , the algebra of  $n \times n$  matrices. Throughout this article, this  $n$  is fixed as the dimension of the spin under consideration, and we fix an orthonormal basis  $\{\psi_\mu\}_{\mu=1}^n$  of  $\mathbb{C}^n$ . We denote the set of all finite subsets in  $\Gamma \subset \mathbb{Z}$  by  $\mathfrak{S}_\Gamma$ . The number of elements in a finite set  $\Lambda \subset \mathbb{Z}$  is denoted by  $|\Lambda|$ . When we talk about intervals in  $\mathbb{Z}$ ,  $[a, b]$  for  $a \leq b$ , means the interval in  $\mathbb{Z}$ , i.e.,  $[a, b] \cap \mathbb{Z}$ . We denote the set of all finite intervals in

$\Gamma$  by  $\mathfrak{I}_\Gamma$ . For each  $z \in \mathbb{Z}$ , we let  $\mathcal{A}_{\{z\}}$  be an isomorphic copy of  $\mathcal{A}$  and for any finite subset  $\Lambda \subset \mathbb{Z}$ ,  $\mathcal{A}_\Lambda = \otimes_{z \in \Lambda} \mathcal{A}_{\{z\}}$  is the local algebra of observables. For finite  $\Lambda$ , the algebra  $\mathcal{A}_\Lambda$  can be regarded as the set of all bounded operators acting on a Hilbert space  $\otimes_{z \in \Lambda} \mathbb{C}^n$ . We use this identification freely. If  $\Lambda_1 \subset \Lambda_2$ , the algebra  $\mathcal{A}_{\Lambda_1}$  is naturally embedded in  $\mathcal{A}_{\Lambda_2}$  by tensoring its elements with the identity. Finally, for an infinite subset  $\Gamma$  of  $\mathbb{Z}$ , the algebra  $\mathcal{A}_\Gamma$  is given as the inductive limit of the algebras  $\mathcal{A}_\Lambda$  with  $\Lambda \in \mathfrak{S}_\Gamma$ . In particular,  $\mathcal{A}_\mathbb{Z}$  is the chain algebra. We denote the set of local observables in  $\Gamma$  by  $\mathcal{A}_\Gamma^{\text{loc}} = \bigcup_{\Lambda \in \mathfrak{S}_\Gamma} \mathcal{A}_\Lambda$ . For  $\omega$  a state on  $\mathcal{A}_\Gamma$  and each finite  $\Lambda \subset \Gamma$ , we denote by  $D_{\omega|_{\mathcal{A}_\Lambda}}$  the density matrix of the restriction  $\omega|_{\mathcal{A}_\Lambda}$ .

For any  $x \in \mathbb{Z}$ , let  $\tau_x$  be the shift operator by  $x$  on  $\mathcal{A}_\mathbb{Z}$ . An interaction is a map  $\Phi$  from  $\mathfrak{S}_\mathbb{Z}$  into  $\mathcal{A}_\mathbb{Z}^{\text{loc}}$  such that  $\Phi(X) \in \mathcal{A}_X$  and  $\Phi(X) = \Phi(X)^*$  for  $X \in \mathfrak{S}_\mathbb{Z}$ . An interaction  $\Phi$  is translation invariant if  $\Phi(X + j) = \tau_j(\Phi(X))$ , for all  $j \in \mathbb{Z}$  and  $X \in \mathfrak{S}_\mathbb{Z}$ . Furthermore, it is of finite range if there exists an  $m \in \mathbb{N}$  such that  $\Phi(X) = 0$ , for  $X$  with diameter larger than  $m$ . In this case, we say that the interaction length of  $\Phi$  is less than or equal to  $m$ . A natural number  $m \in \mathbb{N}$  and an element  $h \in \mathcal{A}_{[0, m-1]}$ , define an interaction  $\Phi_h$  by

$$\Phi_h(X) := \begin{cases} \tau_x(h), & \text{if } X = [x, x + m - 1] \text{ for some } x \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

for  $X \in \mathfrak{S}_\mathbb{Z}$ .

A Hamiltonian associated with  $\Phi$  is a net of self-adjoint operators  $H_\Phi := ((H_\Phi)_\Lambda)_{\Lambda \in \mathfrak{I}_\mathbb{Z}}$  such that

$$(H_\Phi)_\Lambda := \sum_{X \subset \Lambda} \Phi(X). \quad (2)$$

Note that  $(H_\Phi)_\Lambda \in \mathcal{A}_\Lambda$ . Without loss of generality we consider positive interactions i.e.,  $\Phi(X) \geq 0$  for any  $X \in \mathfrak{S}_\mathbb{Z}$ , throughout this article. Let us specify what we mean by gapped with respect to the open boundary conditions Hamiltonian in this paper.

**Definition 1.1.** A Hamiltonian  $H := (H_\Lambda)_{\Lambda \in \mathfrak{I}_\mathbb{Z}}$  associated with a positive translation invariant finite range interaction is *gapped with respect to the open boundary conditions* if there exist  $\gamma > 0$  and  $N_0 \in \mathbb{N}$  such that the difference between the smallest and the next-smallest eigenvalue of  $H_\Lambda$ , is bounded below by  $\gamma$ , for all finite intervals  $\Lambda \subset \mathbb{Z}$  with  $|\Lambda| \geq N_0$ .

In this definition, the smallest eigenvalue can be degenerated in general. Let  $H = (H_\Lambda)$  be a Hamiltonian associated with some positive translation invariant finite range interaction. For a finite interval  $\Lambda$ , a ground state of  $H_\Lambda$  means a state on  $\mathcal{A}_\Lambda$  with support in the smallest eigenvalue space of  $H_\Lambda$ . We denote the set of all ground states of  $H_\Lambda$  on  $\mathcal{A}_\Lambda$  by  $\mathcal{S}_\Lambda(H)$ . For  $\Lambda \in \mathfrak{I}_\Gamma$ , any of the elements in  $\mathcal{S}_\Lambda(H)$  can be extended to a state on  $\mathcal{A}_\Gamma$ , and there exist weak-\* accumulation points of such extensions, in the thermodynamical limit  $\Lambda \rightarrow \Gamma$ . We denote the set of all such accumulation points by  $\mathcal{S}_\Gamma(H)$ .

**Definition 1.2.** We call the quadruplet  $(\{\mathcal{S}(H)\}_{I \in \mathfrak{I}_\Gamma}, \mathcal{S}_{(-\infty, -1]}(H), \mathcal{S}_{[0, \infty)}(H), \mathcal{S}_\mathbb{Z}(H))$  the ground state structure of the Hamiltonian  $H$ .

## 1.2 The parent Hamiltonians

The Hamiltonians we introduce in this paper are parent Hamiltonians of sequence of subspaces satisfying the intersection property. We say that a sequence of subspaces  $\{\mathcal{D}_N\}_{N \in \mathbb{N}}$ ,  $\mathcal{D}_N \subset \bigotimes_{i=0}^{N-1} \mathbb{C}^n$ ,  $N \in \mathbb{N}$ , satisfies the *intersection property*, if there exists an  $m \in \mathbb{N}$ , such that the relation

$$\mathcal{D}_N = \bigcap_{x=0}^{N-m} (\mathbb{C}^n)^{\otimes x} \otimes \mathcal{D}_m \otimes (\mathbb{C}^n)^{\otimes N-m-x}, \quad (3)$$

holds for all  $N \geq m$ . In order to specify the number  $m \in \mathbb{N}$ , we will say that  $\{\mathcal{D}_N\}_{N \in \mathbb{N}}$  satisfies Property (I,  $m$ ) when (3) holds for  $m$  and all  $N \geq m$ . Note that Property (I,  $m$ ) implies Property (I,  $m'$ ) for all  $m' \geq m$ .

**Definition 1.3.** For a sequence of subspaces  $\{\mathcal{D}_N\}_{N \in \mathbb{N}}$ ,  $\mathcal{D}_N \subset \bigotimes_{i=0}^{N-1} \mathbb{C}^n$ ,  $N \in \mathbb{N}$ , we define  $\mathbf{m}_{\{\mathcal{D}_N\}} \in \mathbb{N} \cup \{\infty\}$  by

$$\mathbf{m}_{\{\mathcal{D}_N\}} := \inf \{m \mid \{\mathcal{D}_N\}_N \text{ satisfies Property (I}, m)\}.$$

Let  $\{\mathcal{D}_N\}$  be a sequence of nonzero spaces satisfying Property (I,  $m$ ). Let  $Q_m$  be the orthogonal projection onto the orthogonal complement of  $\mathcal{D}_m$  in  $\bigotimes_{i=0}^{m-1} \mathbb{C}^n$ , and consider the interaction  $\Phi_{Q_m}$  associated with  $Q_m$ . Then, by (3), we see that  $\ker (H_{\Phi_{Q_m}})_{[0, N-1]} = \mathcal{D}_N$  for all  $N \geq m$ . Namely, the ground state spaces of the Hamiltonian  $H_{\Phi_{Q_m}}$  are given by  $\{\mathcal{D}_N\}$ . We shall refer to that particular Hamiltonian as the *parent Hamiltonian* of  $\{\mathcal{D}_N\}$ , and denote this  $\Phi_{Q_m}$  by  $\Phi_{m, \{\mathcal{D}_N\}}$ .

The matrix product formalism gives a way to define sequences of subspaces. Let  $k \in \mathbb{N}$ , and  $\mathbf{v} = (v_1, \dots, v_n) \in (\mathbb{M}_k)^{\times n}$ , an  $n$ -tuple of elements in  $\mathbb{M}_k$ . For  $l \in \mathbb{N}$  and  $\mu^{(l)} = (\mu_0, \mu_1, \dots, \mu_{l-1}) \in \{1, \dots, n\}^{\times l}$ , we use the notation

$$\widehat{v_{\mu^{(l)}}} := v_{\mu_0} v_{\mu_1} \cdots v_{\mu_{l-1}} \in \mathbb{M}_k, \quad \widehat{\psi_{\mu^{(l)}}} := \bigotimes_{i=0}^{l-1} \psi_{\mu_i} \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n. \quad (4)$$

(Recall that  $\{\psi_\mu\}_\mu$  is the fixed CONS of  $\mathbb{C}^n$ .) For an invertible  $R \in \mathbb{M}_k$  and  $\mathbf{v} = (v_1, \dots, v_n) \in (\mathbb{M}_k)^{\times n}$ , we denote by  $R\mathbf{v}R^{-1}$  the  $n$ -tuple given by  $R\mathbf{v}R^{-1} = (Rv_1R^{-1}, \dots, Rv_nR^{-1})$ . We say  $R\mathbf{v}R^{-1}$  is similar to  $\mathbf{v}$  in this case.

**Definition 1.4.** Let  $k \in \mathbb{N}$ , and  $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{M}_k^{\times n}$ . For each  $l \in \mathbb{N}$ , define  $\Gamma_{l, \mathbf{v}}^{(R)} : \mathbb{M}_k \rightarrow \bigotimes_{i=0}^{l-1} \mathbb{C}^n$  by

$$\Gamma_{l, \mathbf{v}}^{(R)}(X) = \sum_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \left( \text{Tr } X (\widehat{v_{\mu^{(l)}}})^* \right) \widehat{\psi_{\mu^{(l)}}}, \quad X \in \mathbb{M}_k, \quad (5)$$

and set  $\mathcal{G}_{l, \mathbf{v}} := \text{Ran } \Gamma_{l, \mathbf{v}}^{(R)} \subset \bigotimes_{i=0}^{l-1} \mathbb{C}^n$ . Furthermore, we denote by  $G_{l, \mathbf{v}}$  the orthogonal projection onto  $\mathcal{G}_{l, \mathbf{v}}$  in  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$ . We set  $h_{m, \mathbf{v}} := 1 - G_{m, \mathbf{v}}$  and  $\Phi_{m, \mathbb{B}} := \Phi_{h_{m, \mathbb{B}}}$ . For the simplicity of the terminology, we use the symbol  $\mathbf{m}_{\mathbf{v}}$  to denote  $\mathbf{m}_{\{\mathcal{G}_{N, \mathbf{v}}\}_N}$ .

With a random choice of  $\mathbf{v}$ , the sequence of subspaces  $\{\mathcal{G}_{l, \mathbf{v}}\}_l$  would not satisfy the intersection property. Furthermore, even if the intersection property is satisfied, the parent Hamiltonian may not be gapped. We need to require some additional conditions to guarantee those properties.

We introduce several notations about  $n$ -tuples of matrices, which we use throughout this paper. For  $k, l \in \mathbb{N}$  and  $\mathbf{v} \in \mathbb{M}_k^{\times n}$ , let  $\mathcal{K}_l(\mathbf{v})$  be the following span of monomials of degree  $l$  in the  $v_\mu$ 's,

$$\mathcal{K}_l(\mathbf{v}) := \text{span} \{v_{\mu_0} v_{\mu_1} \cdots v_{\mu_{l-1}} \mid (\mu_0, \mu_1, \dots, \mu_{l-1}) \in \{1, \dots, n\}^{\times l}\}. \quad (6)$$

For  $k \in \mathbb{N}$  and  $\mathbf{v} \in \mathbb{M}_k^{\times n}$ , we define the completely positive map  $T_{\mathbf{v}} : \mathbb{M}_k \rightarrow \mathbb{M}_k$  by

$$T_{\mathbf{v}}(X) := \sum_{\mu=1}^n v_\mu X v_\mu^*, \quad X \in \mathbb{M}_k. \quad (7)$$

The spectral property of  $T_{\mathbf{v}}$  will play an important role in the analysis. For  $\mathbf{v} \in \mathbb{M}_k^{\times n}$ , and a projection  $p \in \mathbb{M}_k$ , we denote by  $\mathbf{v}_p$  the  $n$ -tuple given by  $\mathbf{v}_p = (pv_1p, pv_2p, \dots, pv_np)$ . For  $p, q \in \mathcal{P}(\mathbb{M}_k)$ , we also set

$$M_{\mathbf{v}, p, q} := \inf \{M \in \mathbb{N} \mid \Gamma_{N, \mathbf{v}}^{(R)} \text{ is injective on } p\mathbb{M}_k q \text{ for all } N \geq M\}. \quad (8)$$

We set  $\mathcal{G}_{N, \mathbf{v}}^{p, q} := \Gamma_{N, \mathbf{v}}^{(R)}(p\mathbb{M}_k q)$  for  $N \in \mathbb{N}$ .

### 1.3 ClassA

In this section we introduce a class of  $n$ -tuples of matrices which we consider in this paper. First we introduce several notations we use repeatedly.

**Definition 1.5.** For  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , we denote by  $\text{Wo}(k_R, k_L)$  the set of all  $\lambda = (\lambda_{-k_R}, \dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots, \lambda_{k_L}) \in \mathbb{C}^{k_R+k_L+1}$  satisfying

$$\begin{aligned} \lambda_0 &= 1, \\ 0 &< |\lambda_{-k_R}| \leq |\lambda_{-k_R+1}| \leq \dots \leq |\lambda_{-1}| < 1, \\ 0 &< |\lambda_{k_L}| \leq |\lambda_{k_L-1}| \leq \dots \leq |\lambda_1| < 1. \end{aligned}$$

For  $\lambda \in \mathbb{C}^{k_R+k_L+1}$ , we define the diagonal matrix  $\Lambda_\lambda := \sum_{i=-k_R}^{k_L} \lambda_i E_{ii}^{(k_R, k_L)}$ .

**Definition 1.6.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ , and  $\omega = (\omega_1, \dots, \omega_n) \in M_{n_0}^{\times n}$ . We say  $\omega$  is primitive if

$$l_\omega := \inf\{l \in \mathbb{N} : \mathcal{K}_{l'}(\omega) = M_{n_0} \text{ for all } l' \geq l\} < \infty.$$

We denote by  $\text{Prim}(n, n_0)$  the set of all primitive  $n$ -tuples  $\omega$  of  $n_0 \times n_0$  matrices. We also denote by  $\text{Prim}_1(n, n_0)$  the set of all primitive  $n$ -tuples  $\omega$  of  $n_0 \times n_0$  matrices with  $r_{T_\omega} = 1$ . Furthermore, we denote by  $\text{Prim}_u(n, n_0)$  the set of all primitive  $n$ -tuples  $\omega$  of  $n_0 \times n_0$  matrices such that  $T_\omega$  is a unital CP map.

**Definition 1.7.** Let  $k_R \in \mathbb{N}$ . We define  $\mathcal{C}^R(k_R)$  by the set of  $k_R$ -tuples  $\mathbb{D} = (D_1, \dots, D_{k_R})$  of  $\text{UT}_{0, k_R+1}$  satisfying the following conditions.

1.  $D_a E_{00}^{(k_R, 0)} = E_{-a, 0}^{(k_R, 0)}$
2. The linear span of  $\{D_a\}_{a=1}^{k_R}$  is a subalgebra of  $\text{UT}_{0, k_R+1}$

Similarly, for  $k_L \in \mathbb{N}$ , we define  $\mathcal{C}^L(k_L)$  by the set of  $k_L$ -tuples  $\mathbb{G} = (G_1, \dots, G_{k_L})$  of  $\text{UT}_{0, k_L+1}$  satisfying the following conditions.:

1.  $E_{00}^{(0, k_L)} G_b = E_{0, b}^{(0, k_L)}$
2. The linear span of  $\{G_b\}_{b=1}^{k_L}$  is a subalgebra of  $\text{UT}_{0, k_L+1}$ .

**Definition 1.8** ( $\mathcal{T}(k_R, k_L)$ ). Let  $(k_R, k_L) \in (\mathbb{N} \cup \{0\})^{\times 2}$ . If  $k_R, k_L \in \mathbb{N}$ , we denote by  $\mathcal{T}(k_R, k_L)$  the set of all quadruplets  $(\lambda, \mathbb{D}, \mathbb{G}, Y)$  with  $\lambda \in \text{Wo}(k_R, k_L)$ ,  $\mathbb{D} \in \mathcal{C}^R(k_R)$ ,  $\mathbb{G} \in \mathcal{C}^L(k_L)$ , and  $Y \in \text{UT}_{0, k_L+k_R+1}$  such that for any  $1 \leq a \leq k_R$  and  $1 \leq b \leq k_L$ ,

$$\Lambda_\lambda I_R^{(k_R, k_L)}(D_a) = \lambda_{-a} I_R^{(k_R, k_L)}(D_a) \Lambda_\lambda, \quad a = 1, \dots, k_R, \quad (9)$$

$$\Lambda_\lambda I_L^{(k_R, k_L)}(G_b) = \lambda_b^{-1} I_L^{(k_R, k_L)}(G_b) \Lambda_\lambda, \quad b = 1, \dots, k_L, \quad (10)$$

$$Y \Lambda_\lambda = \Lambda_\lambda Y, \quad P_R^{(k_R, k_L)} Y P_L^{(k_R, k_L)} = 0, \quad (11)$$

and

$$(\Lambda_\lambda(1+Y))^l I_R^{(k_R, k_L)}(D_a) = \sum_{a'=1}^{k_R} \left\langle f_{-a'}^{(k_R, k_L)}, (\Lambda_\lambda(1+Y))^l f_{-a}^{(k_R, k_L)} \right\rangle I_R^{(k_R, k_L)}(D_{a'}) (\Lambda_\lambda(1+Y))^l, \quad (12)$$

$$I_L^{(k_R, k_L)}(G_b) (\Lambda_\lambda(1+Y))^l = \sum_{b'=1}^{k_L} \left\langle f_b^{(k_R, k_L)}, (\Lambda_\lambda(1+Y))^l f_{b'}^{(k_R, k_L)} \right\rangle (\Lambda_\lambda(1+Y))^l I_L^{(k_R, k_L)}(G_{b'}), \quad (13)$$

for all  $l \in \mathbb{N}$ . If  $k_R = 0$  and  $k_L \in \mathbb{N}$ ,  $\mathcal{T}(0, k_L)$  is the set of all triples  $(\lambda, \mathbb{G}, Y)$  with  $\lambda \in \text{Wo}(k_R, k_L)$ ,  $\mathbb{G} \in \mathcal{C}^L(k_L)$ , and  $Y \in \text{UT}_{0, k_L + k_R + 1}$ , satisfying (10), (11), and (13). If  $k_R \in \mathbb{N}$  and  $k_L = 0$ ,  $\mathcal{T}(k_R, 0)$  is the set of all triples  $(\lambda, \mathbb{D}, Y)$  with  $\lambda \in \text{Wo}(k_R, k_L)$ ,  $\mathbb{D} \in \mathcal{C}^R(k_R)$ , and  $Y \in \text{UT}_{0, k_L + k_R + 1}$ , satisfying (9), (11), and (12). If  $k_R = k_L = 0$ ,  $\mathcal{T}(0, 0)$  consists of a single point  $\lambda_0 = 1 \in \text{Wo}(0, 0)$ .

*Remark 1.9.* As  $\Lambda_\lambda(1 + Y)$  is an upper triangular matrix, the summations on the right hand side of (12) and (13) are actually over  $a \leq a'$  and  $b \leq b'$ .

*Remark 1.10.* For the simplicity of the statements, we write  $\mathbb{D}$  or  $\mathbb{G}$  even when  $k_R = 0$  or  $k_L = 0$ , although they are not defined in these cases. The readers should discard them suitably in each statements.

*Remark 1.11.* Let  $\mathbb{D} \in \mathcal{C}^R(k_R)$  and  $\mathbb{G} \in \mathcal{C}^L(k_L)$ . Note that  $I_R^{(k_R, k_L)}(D_a) = \overline{P_L^{(k_R, k_L)}} I_R^{(k_R, k_L)}(D_a) P_R^{(k_R, k_L)}$ ,  $I_L^{(k_R, k_L)}(G_b) = P_L^{(k_R, k_L)} I_L^{(k_R, k_L)}(G_b) \overline{P_R^{(k_R, k_L)}}$ , because  $D_a$  and  $G_b$  are upper triangular matrices with zero diagonal elements. Therefore, the linear space

$$\mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) := \text{span} \left( \mathbb{I}, \{I_R^{(k_R, k_L)}(D_a)\}_{a=1}^{k_R} \cup \{I_L^{(k_R, k_L)}(G_b)\}_{b=1}^{k_L} \cup \left\{ E_{-a, b}^{(k_R, k_L)} \right\}_{a=1, \dots, k_R, b=1, \dots, k_L} \right) \quad (14)$$

is a subalgebra of  $\text{UT}_{k_L + k_R + 1}$ . (Here, as mentioned in Remark 1.10, discard  $\{I_R^{(k_R, k_L)}(D_a)\}_{a=1}^{k_R}$ ,  $\left\{ E_{-a, b}^{(k_R, k_L)} \right\}_{a=1, \dots, k_R, b=1, \dots, k_L}$ , on the right hand side if  $k_R = 0$ .)

*Remark 1.12.* By (12), (13) and the fact that  $\Lambda_\lambda(1 + Y)$  is an invertible matrix, we observe that

$$\begin{aligned} I_R^{(k_R, k_L)}(D_a) (\Lambda_\lambda(1 + Y))^l &= \sum_{a'=1}^{k_R} \left\langle f_{-a'}^{(k_R, k_L)}, (\Lambda_\lambda(1 + Y))^{-l} f_{-a}^{(k_R, k_L)} \right\rangle (\Lambda_\lambda(1 + Y))^l I_R^{(k_R, k_L)}(D_{a'}), \\ (\Lambda_\lambda(1 + Y))^l I_L^{(k_R, k_L)}(G_b) &= \sum_{b'=1}^{k_L} \left\langle f_b^{(k_R, k_L)}, (\Lambda_\lambda(1 + Y))^{-l} f_{b'}^{(k_R, k_L)} \right\rangle I_L^{(k_R, k_L)}(G_{b'}) (\Lambda_\lambda(1 + Y))^l. \end{aligned} \quad (15)$$

This implies

$$(\Lambda_\lambda(1 + Y))^{-x} \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) (\Lambda_\lambda(1 + Y))^x = \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}), \quad x \in \mathbb{Z}. \quad (16)$$

**Definition 1.13.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$  and  $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$ . We denote by  $\mathfrak{B}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$  the set of all  $n$ -tuples  $\mathbb{B} = (B_1, \dots, B_n) \in \text{M}_{n_0} \otimes (\mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_\lambda(1 + Y))$  satisfying

$$l_{\mathbb{B}} = l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y) := \inf \left\{ l \mid \mathcal{K}_{l'}(\mathbb{B}) = \text{M}_{n_0} \otimes \left( \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) (\Lambda_\lambda(1 + Y))^{l'} \right) \text{ for all } l' \geq l \right\} < \infty, \quad (17)$$

and  $r_{T_{\mathbb{B}}} = 1$ .

**Definition 1.14** (ClassA). Let  $n \geq 2$ . We define

$$\text{ClassA} := \bigcup \{ \mathfrak{B}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y) \mid n_0 \in \mathbb{N}, k_R, k_L \in \mathbb{N} \cup \{0\}, (\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L) \}.$$

*Remark 1.15.* We say  $\mathbb{B}$  belongs to ClassA with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ , when we would like to make them explicitly.

*Remark 1.16.* Recall that the parents Hamiltonians considered in [FNW] are generated by  $\mathbf{v} \in \text{Prim}(n, n_0)$ , i.e., the condition  $\mathcal{K}_l(\mathbf{v}) = \text{M}_{n_0}$  for large  $l$ , is required. This is what we call the injectivity condition. This corresponds to the case  $k_L = k_R = 0$  in our setting.

**Example 1.17.** Let  $n_0 \in \mathbb{N}$ , and  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . We fix  $0 < \kappa < 1$ , and set  $\boldsymbol{\lambda} = (\lambda_i)_{i=-k_R}^{k_L}$  and  $\mathbf{r} = (r_\alpha)_{\alpha=1}^{n_0}$  by

$$\begin{aligned} r_\alpha &:= \kappa^{\alpha-1}, \quad \text{for } \alpha = 1, \dots, n_0, \\ \lambda_j &:= \kappa^{|j|n_0}, \quad \text{for } j = -k_R, \dots, -1, 0, 1, \dots, k_L. \end{aligned}$$

We also set

$$V_R := \sum_{j=-k_R}^{-1} E_{j,j+1}^{(k_R,0)}, \quad V_L := \sum_{j=0}^{k_L-1} E_{j,j+1}^{(0,k_L)},$$

and define  $\mathbb{D}$  and  $\mathbb{G}$  by

$$D_a := V_R^a, \quad a = 1, \dots, k_R, \quad G_b := V_L^b, \quad b = 1, \dots, k_L.$$

It is easy to check  $(\boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, 0) \in \mathcal{T}(k_R, k_L)$ . Set

$$\eta := \sum_{\alpha=2}^{(n_0)} \chi_\alpha^{(n_0)}.$$

We define  $\mathbb{B} = (B_\mu)$  by

$$\begin{aligned} B_1 &:= \Lambda_{\mathbf{r}} \otimes \Lambda_{\boldsymbol{\lambda}}, \\ B_2 &:= \left( \left| \chi_1^{(n_0)} \right\rangle \langle \eta| + |\eta\rangle \left\langle \chi_1^{(n_0)} \right| \right) \otimes \Lambda_{\boldsymbol{\lambda}} + \Lambda_{\mathbf{r}} \otimes \left( I_R^{(k_R, k_L)}(V_R) + I_L^{(k_R, k_L)}(V_L) \right) \Lambda_{\boldsymbol{\lambda}}, \\ B_\mu &:= 0, \quad \mu \geq 3. \end{aligned}$$

We claim  $\mathbb{B} \in \mathfrak{B}(n, n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, 0)$ . From the definition, we have  $B_\mu \in M_{n_0} \otimes (\mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_{\boldsymbol{\lambda}})$ , for each  $\mu = 1, \dots, n$ . This fact and Remark 1.11, Remark 1.12 imply that  $\mathcal{K}_l(\mathbb{B}) \subset M_{n_0} \otimes (\mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_{\boldsymbol{\lambda}}^l)$  holds for all  $l \in \mathbb{N}$ . Now we prove the opposite inclusion for  $l$  large enough. We use the argument in Lemma 5.2 [BO]. Let  $l \in \mathbb{N}$  and  $0 \leq j \leq l-1$ . By the expansion, we see that

$$B_1^{l-1-j} B_2 B_1^j = \sum_{\alpha=2}^{n_0} \left( r_\alpha^j e_{1\alpha}^{(n_0)} + r_\alpha^{l-1-j} e_{\alpha 1}^{(n_0)} \right) \otimes \Lambda_{\boldsymbol{\lambda}}^l + \kappa^{(j-l+1)n_0} \Lambda_{\mathbf{r}}^l \otimes I_L^{(k_R, k_L)}(V_L) \Lambda_{\boldsymbol{\lambda}}^l + \kappa^{(l-1-j)n_0} \Lambda_{\mathbf{r}}^l \otimes I_R^{(k_R, k_L)}(V_R) \Lambda_{\boldsymbol{\lambda}}^l.$$

As the numbers  $\{r_\alpha = \kappa^{\alpha-1}\}_{\alpha=2}^{n_0}$ ,  $\{r_\alpha^{-1} = \kappa^{-\alpha+1}\}_{\alpha=2}^{n_0}$ ,  $\{\kappa^{n_0}\}$ ,  $\{\kappa^{-n_0}\}$  are distinct, by the argument of Lemma 5.2 [BO], we conclude that for  $l$  large enough, all of

$$e_{1\alpha}^{(n_0)} \otimes \Lambda_{\boldsymbol{\lambda}}^l, \quad e_{\alpha 1}^{(n_0)} \otimes \Lambda_{\boldsymbol{\lambda}}^l, \quad \Lambda_{\mathbf{r}}^l \otimes I_L^{(k_R, k_L)}(V_L) \Lambda_{\boldsymbol{\lambda}}^l, \quad \Lambda_{\mathbf{r}}^l \otimes I_R^{(k_R, k_L)}(V_R) \Lambda_{\boldsymbol{\lambda}}^l$$

belong to  $\mathcal{K}_l(\mathbb{B})$ . Multiplying these terms each other, we conclude that  $M_{n_0} \otimes (\mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) \Lambda_{\boldsymbol{\lambda}}^l) \subset \mathcal{K}_l(\mathbb{B})$  holds for all  $l$  large enough.

## 1.4 The main result

In this part I, we study the ground state structure of the Hamiltonians given by  $\mathbb{B} \in \text{ClassA}$ . Our Hamiltonian is the Hamiltonian  $H_{\Phi_{m, \mathbb{B}}}$  given by the subspaces  $\mathcal{G}_{m, \mathbb{B}}$  via the formula (1) with  $h = 1 - G_{m, \mathbb{B}}$  for  $\mathbb{B} \in \text{ClassA}$ .

**Theorem 1.18.** Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y)$ .

Then  $m_{\mathbb{B}} \leq 2l_{\mathbb{B}}(n, n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y) < \infty$  and for any  $m \geq \max\{2l_{\mathbb{B}}(n, n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y), \frac{\log(n_0^2(k_L+1)(k_R+1)+1)}{\log n}\}\}$ , we have the followings.

(i)  $\ker (H_{\Phi_{m,\mathbb{B}}})_{[0,N-1]} = \mathcal{G}_{N,\mathbb{B}}$  and  $\dim \ker (H_{\Phi_{m,\mathbb{B}}})_{[0,N-1]} = n_0^2(k_L + 1)(k_R + 1)$ , for  $N \geq m$ .

(ii) There exist  $\gamma_m > 0$  and  $N_m \in \mathbb{N}$  such that

$$\gamma_m (1 - G_{N,\mathbb{B}}) \leq (H_{\Phi_{m,\mathbb{B}}})_{[0,N-1]}, \text{ for all } N \geq N_m.$$

(iii)  $\mathcal{S}_{\mathbb{Z}}(H_{\Phi_{m,\mathbb{B}}})$  consists of a unique state  $\omega_{\mathbb{B},\infty}$  on  $\mathcal{A}_{\mathbb{Z}}$ .

(iv) There exist  $0 < C_{\mathbb{B}}$ ,  $0 < s_{\mathbb{B}} < 1$ ,  $N_{\mathbb{B}} \in \mathbb{N}$ , and states  $\omega_{R,\mathbb{B}} \in \mathcal{S}_{[0,\infty)}(H_{\Phi_{m,\mathbb{B}}})$ , and  $\omega_{L,\mathbb{B}} \in \mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m,\mathbb{B}}})$ , such that

$$\begin{aligned} \left| \frac{\text{Tr}_{[0,N-1]}(G_{N,\mathbb{B}}A)}{\text{Tr}_{[0,N-1]}(G_{N,\mathbb{B}})} - \omega_{R,\mathbb{B}}(A) \right| &\leq C_{\mathbb{B}} s_{\mathbb{B}}^{N-l} \|A\|, \\ \left| \frac{\text{Tr}_{[0,N-1]}(G_{N,\mathbb{B}}\tau_{N-l}(A))}{\text{Tr}_{[0,N-1]}(G_{N,\mathbb{B}})} - \omega_{L,\mathbb{B}} \circ \tau_{-l}(A) \right| &\leq C_{\mathbb{B}} s_{\mathbb{B}}^{N-l} \|A\|, \end{aligned} \quad (18)$$

for all  $l \in \mathbb{N}$ ,  $A \in \mathcal{A}_{[0,l-1]}$ , and  $N \geq \max\{l, N_{\mathbb{B}}\}$ , and

$$\begin{aligned} \inf \{ \sigma(\omega_{R,\mathbb{B}}|_{\mathcal{A}_{[0,l-1]}}) \setminus \{0\} \mid l \in \mathbb{N} \} &> 0, \\ \inf \{ \sigma(\omega_{L,\mathbb{B}}|_{\mathcal{A}_{[-l,-1]}}) \setminus \{0\} \mid l \in \mathbb{N} \} &> 0. \end{aligned} \quad (19)$$

(v) For any  $\psi \in \mathcal{S}_{[0,\infty)}(H_{\Phi_{m,\mathbb{B}}})$  (resp.  $\psi \in \mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m,\mathbb{B}}})$ ), there exists an  $l_{\psi} \in \mathbb{N}$  such that  $\|\psi - \psi \circ \tau_{l_{\psi}}\| < 2$  (resp.  $\|\psi - \psi \circ \tau_{-l_{\psi}}\| < 2$ ).

(vi)  $\mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m,\mathbb{B}}})$  and  $\mathcal{S}_{[0,+\infty)}(H_{\Phi_{m,\mathbb{B}}})$  are convex sets and there exist affine bijections

$$\Xi_L : \mathfrak{E}_{n_0(k_L+1)} \rightarrow \mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m,\mathbb{B}}}), \quad \Xi_R : \mathfrak{E}_{n_0(k_R+1)} \rightarrow \mathcal{S}_{[0,+\infty)}(H_{\Phi_{m,\mathbb{B}}}).$$

(vii) There exist  $C'_{\mathbb{B}} > 0$ , and  $0 < s'_{\mathbb{B}} < 1$  such that

$$\begin{aligned} |\psi \circ \tau_N(A) - \omega_{\mathbb{B},\infty}(A)| &\leq C'_{\mathbb{B}} s'_{\mathbb{B}}{}^N \|A\|, \\ (\text{resp. } |\psi \circ \tau_{-N}(A) - \omega_{\mathbb{B},\infty}(A)| &\leq C'_{\mathbb{B}} s'_{\mathbb{B}}{}^N \|A\|, ) \end{aligned} \quad (20)$$

for all  $A \in \mathcal{A}_{[0,\infty)}$  and  $\psi \in \mathcal{S}_{[0,\infty)}(H_{\Phi_{m,\mathbb{B}}})$  (resp. for all  $A \in \mathcal{A}_{(-\infty,-1]}$  and  $\psi \in \mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m,\mathbb{B}}})$ ), and  $N \in \mathbb{N}$ .

(viii) Any element in  $\mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m,\mathbb{B}}})$  or  $\mathcal{S}_{[0,+\infty)}(H_{\Phi_{m,\mathbb{B}}})$  is a factor state.

(ix)  $\omega_{\mathbb{B},\infty}$  satisfies the exponential decay of correlations.

This article is organized as follows: In Section 2, we introduce a sufficient condition for an  $n$ -tuple  $\mathbf{v}$  to have a gapped parent Hamiltonian, in rather general setting. Applying the results of Section 2, we study the properties of  $H_{\Phi_{m,\mathbb{B}}}$  for  $\mathbb{B} \in \text{ClassA}$  in Section 3.

## 2 The intersection property and the spectral gap

In this section, we introduce a sufficient condition for the MPS Hamiltonians to be gapped. We first introduce a set of conditions on sequences of subspaces.

**Definition 2.1** (*Condition 1*). Let  $n \in \mathbb{N}$ . Let  $\mathcal{D}_l$  be a nonzero subspace of  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$  given for each  $l \in \mathbb{N}$ . We denote by  $G_l$  the orthogonal projection onto  $\mathcal{D}_l$  in  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$  for each  $l \in \mathbb{N}$ . We say the sequence of subspaces  $\{\mathcal{D}_l\}_{l \in \mathbb{N}}$  satisfies the *Condition 1* if there exists  $m_0, l_0 \in \mathbb{N}$  such that



- (i)  $\{\mathcal{D}_l\}_{l \in \mathbb{N}}$  satisfies Property (I,  $m_0$ ), and
- (ii) for any  $l_0 \leq l$ , there exists  $0 < \varepsilon_l < \frac{1}{\sqrt{l}}$  such that

$$\|(1_{[0, N-l]} \otimes G_l)(G_N \otimes 1_{\{N\}} - G_{N+1})\| < \varepsilon_l,$$

for all  $N \geq 2l$ .

We say  $\{\mathcal{D}_l\}_{l \in \mathbb{N}}$  satisfies the *Condition 1* for  $(m_0, l_0)$  when we would like to specify the numbers.

The following theorem is a special version of Theorem 3 in [N].

**Theorem 2.2** ([N]). *Let  $\{\mathcal{D}_l\}_{l \in \mathbb{N}}$  be a sequence of nonzero vector spaces such that  $\mathcal{D}_l \subset \bigotimes_{i=0}^{l-1} \mathbb{C}^n$ , and  $G_l$  the orthogonal projections onto  $\mathcal{D}_l$  in  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$ . For  $m, N \in \mathbb{N}$  with  $m \leq N$ , we set*

$$H_N^m := \sum_{0 \leq x \leq N-m} \tau_x (1 - G_m).$$

*Suppose that  $\{\mathcal{D}_l\}_l$  satisfies the Condition 1 for  $(m_0, l_0)$ . Then for all  $m_0 \leq m$ ,*

*(1)  $\ker H_N^m = \mathcal{D}_N$ , for all  $N \geq m$ , and*

*(2)*

$$\frac{\gamma_{l,m}}{l+2} \left(1 - \varepsilon_l \sqrt{l}\right)^2 (1 - G_N) \leq H_N^m,$$

*for all  $l$  with  $\max\{l_0, m\} \leq l$ , and  $N$  with  $2l + 1 \leq N$ , where*

$$\gamma_{l,m} = \min \{d_{\mathbb{R}}(\sigma(H_l^m) \setminus \{0\}, 0), d_{\mathbb{R}}(\sigma(H_{2l}^m) \setminus \{0\}, 0)\}.$$

In this section, we give a criterion for this in the MPS formalism. We introduce a set of conditions. Recall the definition (6) of  $\mathcal{K}_l(\mathbf{v})$ .

**Definition 2.3** (*Condition 2*). For  $n, k \in \mathbb{N}$ , projections  $p, q \in \mathcal{P}(\mathbb{M}_k)$ , and  $n$ -tuple of  $k \times k$  matrices  $\mathbf{v} \in \mathbb{M}_k^{\times n}$ , we say that the pentad  $(n, k, p, q, \mathbf{v})$  satisfies *Condition 2* if the following conditions are satisfied.

- (i)  $pq = qp \neq 0$ .
- (ii)  $v_\mu p = p v_\mu p$  for all  $\mu = 1, \dots, n$ .
- (iii)  $q v_\mu = q v_\mu q$  for all  $\mu = 1, \dots, n$ .
- (iv) There exist constants  $c_{pq} > 0$ ,  $0 < s_{pq} < 1$ , a positive linear functional  $\varphi_{pq}$  on  $\mathbb{M}_k$  and a positive element  $e_{pq} \in \mathbb{M}_k$  with  $s(\varphi_{pq}) = s(e_{pq}) = pq$ , such that

$$\|T_{\mathbf{v}_{pq}}^N(A) - \varphi_{pq}(A)e_{pq}\| \leq c_{pq}s_{pq}^N \|A\|, \quad \text{for all } A \in \mathbb{M}_k, \text{ and } N \in \mathbb{N}.$$

- (v) There exist constants  $c_{\bar{q}} > 0$ ,  $0 < s_{\bar{q}} < 1$  such that

$$\|T_{\mathbf{v}_{\bar{q}}}^N(A)\| \leq c_{\bar{q}}s_{\bar{q}}^N \|A\|, \quad \text{for all } A \in \mathbb{M}_k \text{ and } N \in \mathbb{N}.$$

- (vi) There exist constants  $c_{\bar{p}} > 0$ ,  $0 < s_{\bar{p}} < 1$  such that

$$\|T_{\mathbf{v}_{\bar{p}}}^N(A)\| \leq c_{\bar{p}}s_{\bar{p}}^N \|A\|, \quad \text{for all } A \in \mathbb{M}_k \text{ and } N \in \mathbb{N}.$$

(vii) For any  $\eta \in q\mathbb{C}^k$  with  $pq\eta \neq 0$ , there exists  $l_\eta \in \mathbb{N}$  such that  $(\mathcal{K}_{l_\eta}(\mathbf{v}_q))^* \eta = q\mathbb{C}^k$ .

(viii) For any  $\xi \in p\mathbb{C}^k$  with  $pq\xi \neq 0$ , there exists  $l'_\xi \in \mathbb{N}$  such that  $\mathcal{K}_{l'_\xi}(\mathbf{v}_p)\xi = p\mathbb{C}^k$ .

**Definition 2.4** (*Condition 3*). Let  $n, k \in \mathbb{N}$ ,  $p, q \in \mathcal{P}(\mathbb{M}_k)$  and  $\mathbf{v} \in \mathbb{M}_k^{\times n}$ . We say that the pentad  $(n, k, p, q, \mathbf{v})$  satisfies the Condition 3 for  $m_1 \in \mathbb{N}$  if for all  $N \geq m_1$ ,  $\dim \mathcal{K}_N(\mathbf{v}) = (\text{rank } p)(\text{rank } q)$ .

**Definition 2.5** (*Condition 4*). Let  $n, k \in \mathbb{N}$ , and  $\mathbf{v} \in \mathbb{M}_k^{\times n}$ . For  $m_2, m_3 \in \mathbb{N}$ , we say that the triple  $(n, k, \mathbf{v})$  satisfies the Condition 4 for  $(m_2, m_3)$ , if there exists an invertible  $X_{m_2} \in \mathcal{K}_{m_2}(\mathbf{v})$  such that  $X_{m_2}^{-1}\mathcal{K}_{N+m_2}(\mathbf{v}) \subset \mathcal{K}_N(\mathbf{v})$  for all  $N \geq m_3$ . When we would like to specify  $X_{m_2}$ , we say the triple  $(n, k, \mathbf{v})$  satisfies the Condition 4 for  $(m_2, m_3)$  with respect to  $X_{m_2}$ .

Here is the main Proposition of this section.

**Proposition 2.6.** Let  $n, k, m_1, m_2, m_3 \in \mathbb{N}$ ,  $p, q \in \mathcal{P}(\mathbb{M}_k)$ , and  $\mathbf{v} = (v_\mu)_{\mu=1}^n \in \mathbb{M}_k^{\times n}$ . Suppose that the pentad  $(n, k, p, q, \mathbf{v})$  satisfies the Condition 2, and the Condition 3 for  $m_1$ . Furthermore, assume that the triple  $(n, k, \mathbf{v})$  satisfies the Condition 4 for  $(m_2, m_3)$ . Then  $M_{\mathbf{v}, p, q} < \infty$  and  $\{\mathcal{G}_{l, \mathbf{v}}\}_{l \in \mathbb{N}}$  satisfies the Condition 1. Here,  $m_0 \in \mathbb{N}$  of Condition 1 can be taken  $m_0 = m_2 + m_3$ .

As we mentioned in the introduction, the spectral property of  $T_{\mathbf{v}}$  is important for the analysis of the spectral gap. In Subsection 2.1, we consider the spectral property of  $T_{\mathbf{v}}$  when  $(n, k, p, q, \mathbf{v})$  satisfies Condition 2. The intersection property is considered in subsection 2.2. The Condition 2 and Condition 3 imply the bijectivity of  $\Gamma_{N, \mathbf{v}}^{(R)}|_{p\mathbb{M}_k q}$ , which is proven in subsection 2.3. The important input to show the spectral gap is an estimate of overlaps of spectral projections. This is done in subsection 2.4.

## 2.1 Spectral analysis of CP maps

**Definition 2.7.** Let  $T$  be a CP map on  $\mathbb{M}_k$ . Let  $0 < s < 1$ ,  $\varphi$  a state on  $\mathbb{M}_k$ , and  $e \in \mathbb{M}_{k+}$ . We say  $T$  satisfies the Spectral Property II with respect to  $(s, e, \varphi)$  if

- (1)  $r_T = 1$  and 1 is a non-degenerate eigenvalue of  $T$ ,
- (2)  $\sigma(T) \setminus \{1\} \subset B_s(0)$ ,
- (3)  $e$  is a  $T$ -invariant positive element and  $e = P_{\{1\}}^T(1)$ ,
- (4)  $\varphi$  is  $T$ -invariant,
- (5)  $P_{\{1\}}^T(\cdot) = \varphi(\cdot)e$ ,
- (6) for any  $s \leq s' < 1$ , we have

$$\|T^N(A) - \varphi(A)e\| \leq (s')^{N+1} \sup_{|z|=s'} \|(z - T)^{-1}\| \|A\|, \quad \text{for all } N \in \mathbb{N}, A \in \mathbb{M}_k.$$

*Remark 2.8.* The conditions above are redundant. For example, (6) follows from (1)-(5). However, we leave them for the convenience.

In this subsection we prove the following Lemma.

**Lemma 2.9.** Let  $n, k \in \mathbb{N}$ ,  $p, q \in \mathcal{P}(\mathbb{M}_k)$ , and  $\mathbf{v} \in \mathbb{M}_k^{\times n}$ . Assume that the pentad  $(n, k, p, q, \mathbf{v})$  satisfies Condition 2. Then, there exist a constant  $0 < s_{\mathbf{v}} < 1$ , a state  $\varphi_{\mathbf{v}}$ , and a positive element  $e_{\mathbf{v}} \in \mathbb{M}_{k+}$ , such that

- (1)  $T_{\mathbf{v}}$  satisfies the Spectral Property II with respect to  $(s_{\mathbf{v}}, e_{\mathbf{v}}, \varphi_{\mathbf{v}})$ , and
- (2)  $s(e_{\mathbf{v}}) = p, \quad s(\varphi_{\mathbf{v}}) = q$ .

*Remark 2.10.* We call  $(s_v, e_v, \varphi_v)$  the triple associated with  $(n, k, p, q, v)$ . From now on, if we write  $(s_v, e_v, \varphi_v)$ , it means this triple. Furthermore,  $\rho_v$  denotes the density matrix of  $\varphi_v$ .

As  $s(e_v) = p$  and  $s(\varphi_v) = q$ , there exist  $x_v \in (pM_k p)_+$  and  $y_v \in (qM_k q)_+$  such that  $e_v x_v = x_v e_v = p$  and  $\rho_v y_v = y_v \rho_v = q$ . We set  $a_v := \|x_v\|^{-1}$  and  $c_v := \|y_v\|^{-1}$ .

The following numbers will be used.

$$\begin{aligned} \tilde{E}_v(N) &:= k^2 \left\| T_v^N \left( 1 - P_{\{1\}}^{T_v} \right) \right\|, \quad E_v(N) := (a_v c_v)^{-1} \tilde{E}_v(N), \quad N \in \mathbb{N}, \\ F_v &:= \sup_{N \in \mathbb{N}} \left\| T_v^N \left( 1 - P_{\{1\}}^{T_v} \right) \right\| + \|e_v\|, \quad L_v := \inf \left\{ L \in \mathbb{N} \mid \sup_{N \geq L} E_v(N) < \frac{1}{2} \right\}. \end{aligned} \quad (21)$$

From Lemma 2.9, we know that  $\lim_{N \rightarrow \infty} \left\| T_v^N \left( 1 - P_{\{1\}}^{T_v} \right) \right\| = 0$ . Hence we have  $\lim_{N \rightarrow \infty} \tilde{E}_v(N) = \lim_{N \rightarrow \infty} E_v(N) = 0$ ,  $F_v < \infty$  and  $L_v \in \mathbb{N}$ . By the definition, we have

$$\sup_{N \in \mathbb{N}} \|T_v^N\| \leq F_v. \quad (22)$$

With  $e_v$  and  $\varphi_v$ , we define a quasi-linear form  $\langle \cdot, \cdot \rangle_v$  on  $M_k$  by

$$\langle A, B \rangle_v := \varphi_v(A^* e_v B), \quad A, B \in M_k.$$

Note that for any  $X \in pM_k q$ ,

$$\|X\|_2^2 \leq (a_v c_v)^{-1} \langle X, X \rangle_v. \quad (23)$$

From this, we see that  $\langle \cdot, \cdot \rangle_v$  is an inner product on  $pM_k q$ . For any  $Z \in M_k$ , we have

$$\langle Z, Z \rangle_v^{\frac{1}{2}} = \langle pZq, pZq \rangle_v^{\frac{1}{2}} = \sup \{ |\langle pZq, Y \rangle_v| \mid Y \in pM_k q, \langle Y, Y \rangle_v = 1 \}. \quad (24)$$

For the first equality, we used the fact that  $e_v \in pM_k p$  and  $\rho_v \in qM_k q$ .

To prove Lemma 2.9, first we note the following basic properties.

**Lemma 2.11.** *Let  $n, k \in \mathbb{N}$ ,  $p \in \mathcal{P}(M_k)$  and  $v \in M_k^{\times n}$  such that  $v_\mu p = p v_\mu p$ ,  $\mu = 1, \dots, n$ . Then,*

1. *for all  $N \in \mathbb{N}$ ,  $\mu^{(N)} \in \{1, \dots, n\}^{\times N}$  and  $A \in M_k$ , we have*

$$\begin{aligned} p(\widehat{v_{\mu^{(N)}}})^* \bar{p} &= 0, \quad \bar{p}(\widehat{v_{\mu^{(N)}}})^* \bar{p} = (\widehat{v_{\mu^{(N)}}})^* \bar{p}, \\ \sum_{\mu^{(N)} \in \{1, \dots, n\}^{\times N}} \bar{p}(\widehat{v_{\mu^{(N)}}}) A (\widehat{v_{\mu^{(N)}}})^* \bar{p} &= T_{\bar{v}_p}^N(A), \\ \sum_{\mu^{(N)} \in \{1, \dots, n\}^{\times N}} (\widehat{v_{\mu^{(N)}}}) p A p (\widehat{v_{\mu^{(N)}}})^* &= T_{v_p}^N(A), \end{aligned}$$

2. *for any  $\eta \in \mathbb{C}^k$  and  $N \in \mathbb{N}$ , we have*

$$\sum_{\mu^{(N)} \in \{1, \dots, n\}^{\times N}} \left\| \bar{p}(\widehat{v_{\mu^{(N)}}})^* p \eta \right\|^2 \leq \left( \sum_{m=1}^N \left( \text{Tr } T_{\bar{v}_p}^{N-m}(1) \right)^{\frac{1}{2}} \left\langle \eta, T_{v_p}^{m-1}(1) \eta \right\rangle^{\frac{1}{2}} \right)^2 \sum_{\mu=1}^n \|v_\mu\|^2$$

3. *for any  $A \in M_k$  and  $N \in \mathbb{N}$ ,*

$$\begin{aligned} &\|T_v^N(A) - pT_{\bar{v}_p}^N(A)p\| \\ &\leq 2\|A\| \left\| T_{\bar{v}_p}^N(1) \right\|^{\frac{1}{2}} \\ &\left( \sup_{M \in \mathbb{N}} \left\| T_{v_p}^M \right\|^{\frac{1}{2}} + \sup_{M \in \mathbb{N}} \left\| T_{\bar{v}_p}^M \right\|^{\frac{1}{2}} + \left( \sum_{m=1}^N \left( \text{Tr } T_{\bar{v}_p}^{N-m}(1) \right)^{\frac{1}{2}} \left\| T_{v_p}^{m-1}(1) \right\|^{\frac{1}{2}} \right) \left( \sum_{\mu=1}^n \|v_\mu\|^2 \right)^{\frac{1}{2}} \right), \end{aligned}$$

4.

$$\begin{aligned} \sup_{M \in \mathbb{N}} \|T_{\mathbf{v}}^M\| &\leq \sup_{M \in \mathbb{N}} \|T_{\mathbf{v}_p}^M\| + \sup_{M \in \mathbb{N}} \|T_{\mathbf{v}_{\bar{p}}}^M\| + 2 \sup_{M \in \mathbb{N}} \left( \sum_{m=1}^M \left( \text{Tr } T_{\mathbf{v}_{\bar{p}}}^{M-m}(1) \right)^{\frac{1}{2}} \|T_{\mathbf{v}_p}^{m-1}(1)\|^{\frac{1}{2}} \right) \left( \sum_{\mu=1}^n \|v_{\mu}\|^2 \right)^{\frac{1}{2}} \|T_{\mathbf{v}_{\bar{p}}}^M\|^{\frac{1}{2}} \\ &\quad + \sup_{M \in \mathbb{N}} \left( \sum_{m=1}^M \left( \text{Tr } T_{\mathbf{v}_{\bar{p}}}^{M-m}(1) \right)^{\frac{1}{2}} \|T_{\mathbf{v}_p}^{m-1}(1)\|^{\frac{1}{2}} \right)^2 \sum_{\mu=1}^n \|v_{\mu}\|^2. \end{aligned}$$

See Section B for the proof.

**Lemma 2.12.** *Let  $n, k \in \mathbb{N}$ ,  $p \in \mathcal{P}(\mathbf{M}_k)$  and  $\mathbf{v} \in \mathbf{M}_k^{\times n}$ , and suppose that (ii),(vi) of Condition 2 is satisfied for this  $(n, k, p, \mathbf{v})$ . Furthermore, assume that there exist constants  $c_p > 0$ ,  $0 < s_p < 1$ , a positive linear functional  $\varphi_p$  on  $\mathbf{M}_k$  and a positive element  $e_p \in \mathbf{M}_k$  such that  $s(\varphi_p), s(e_p) \leq p$ ,  $\varphi_p(e_p) \neq 0$  and*

$$\|T_{\mathbf{v}_p}^N(A) - \varphi_p(A)e_p\| \leq c_p s_p^N \|A\|, \quad \text{for all } A \in \mathbf{M}_k, \text{ and } N \in \mathbb{N}. \quad (25)$$

Then, there exist a positive linear functional  $\varphi_{\mathbf{v},p}^{(r)}$  and a constant  $0 < s_{\mathbf{v},p}^{(r)} < 1$  such that  $\varphi_{\mathbf{v},p}^{(r)}(e_p) = 1$ , and  $T_{\mathbf{v}}$  satisfies the Spectral Property II with respect to  $(s_{\mathbf{v},p}^{(r)}, \varphi_{\mathbf{v},p}^{(r)}(1)e_p, (\varphi_{\mathbf{v},p}^{(r)}(1))^{-1} \varphi_{\mathbf{v},p}^{(r)})$ . If furthermore for any  $\eta \in \mathbb{C}^k$  with  $p\eta \neq 0$ , there exists an  $l_{\eta} \in \mathbb{N}$  such that  $(K_{l_{\eta}}(\mathbf{v}))^* \eta = \mathbb{C}^k$ . then  $\varphi_{\mathbf{v},p}^{(r)}$  is faithful.

**Proof.** By (ii) of the Condition 2, we can apply Lemma 2.11. By (vi) of the Condition 2 and (25), we have  $\sup_{M \in \mathbb{N}} \|T_{\mathbf{v}_p}^M\| < \infty$  and  $\sup_{M \in \mathbb{N}} \|T_{\mathbf{v}_{\bar{p}}}^M\| < \infty$ . Furthermore, we have

$$\sup_{N \in \mathbb{N}} \left( \sum_{m=1}^N \left( \text{Tr } T_{\mathbf{v}_{\bar{p}}}^{N-m}(1) \right)^{\frac{1}{2}} \|T_{\mathbf{v}_p}^{m-1}(1)\|^{\frac{1}{2}} \right) \leq \sup_{M \in \mathbb{N}} \|T_{\mathbf{v}_p}^M\|^{\frac{1}{2}} \sup_{N \in \mathbb{N}} \left( \sum_{m=1}^N \sqrt{k} c_p^{\frac{1}{2}} s_p^{\frac{1}{2}(N-m)} \right) < \infty.$$

Therefore, by 4 of Lemma 2.11,  $\sup_{M \in \mathbb{N}} \|T_{\mathbf{v}}^M\| < \infty$ . Furthermore, by (vi) of the Condition 2 and Lemma 2.11 3, we have

$$\lim_{l \rightarrow \infty} \|T_{\mathbf{v}}^l(\cdot) - pT_{\mathbf{v}}^l(\cdot)p\| = 0. \quad (26)$$

For any  $\varepsilon > 0$ , choose  $l_0 \in \mathbb{N}$  such that  $(\|T_{\mathbf{v}}^{l_0}(\cdot) - pT_{\mathbf{v}}^{l_0}(\cdot)p\| \sup_{L \in \mathbb{N}} \|T_{\mathbf{v}}^L\|) < \frac{\varepsilon}{4}$  and  $c_p s_p^{l_0} \sup_{L \in \mathbb{N}} \|T_{\mathbf{v}}^L\| < \frac{\varepsilon}{4}$ . For any  $N, M \geq 2l_0$ , using 1, 3 of Lemma 2.11 and (25) we have

$$\begin{aligned} &\|T_{\mathbf{v}}^N(A) - T_{\mathbf{v}}^M(A)\| \\ &\leq \|T_{\mathbf{v}}^{N-l_0}(T_{\mathbf{v}}^{l_0}(A) - pT_{\mathbf{v}}^{l_0}(A)p)\| + \|T_{\mathbf{v}}^{M-l_0}(T_{\mathbf{v}}^{l_0}(A) - pT_{\mathbf{v}}^{l_0}(A)p)\| \\ &\quad + \|T_{\mathbf{v}_p}^{N-l_0}(pT_{\mathbf{v}}^{l_0}(A)p) - \varphi_p(pT_{\mathbf{v}}^{l_0}(A)p)e_p\| + \|T_{\mathbf{v}_p}^{M-l_0}(pT_{\mathbf{v}}^{l_0}(A)p) - \varphi_p(pT_{\mathbf{v}}^{l_0}(A)p)e_p\| < \varepsilon \|A\|. \end{aligned}$$

Hence  $\{T_{\mathbf{v}}^N\}$  is a Cauchy sequence and has a limit  $T_{\mathbf{v}}^{\infty}$ .

Clearly we have  $T_{\mathbf{v}}^{\infty} \circ T_{\mathbf{v}} = T_{\mathbf{v}}^{\infty}$  and  $T_{\mathbf{v}}^{\infty}$  is positive. Furthermore, for any  $A \in \mathbf{M}_k$ ,

$$T_{\mathbf{v}}^{\infty}(A) = \lim_{N \rightarrow \infty} T_{\mathbf{v}}^{2N}(A) = \lim_{N \rightarrow \infty} T_{\mathbf{v}}^N(pT_{\mathbf{v}}^N(A)p) = \lim_{N \rightarrow \infty} T_{\mathbf{v}_p}^N(pT_{\mathbf{v}}^N(A)p) = \lim_{N \rightarrow \infty} \varphi_p(pT_{\mathbf{v}}^N(A)p)e_p \in \mathbb{C}e_p.$$

Therefore, there exists a positive linear functional  $\varphi_{\mathbf{v},p}^{(r)}$  such that  $T_{\mathbf{v}}^{\infty}(\cdot) = \varphi_{\mathbf{v},p}^{(r)}(\cdot)e_p$ . By the  $T_{\mathbf{v}}$ -invariance of  $T_{\mathbf{v}}^{\infty}$ ,  $\varphi_{\mathbf{v},p}^{(r)}$  is  $T_{\mathbf{v}}$ -invariant. Note that

$$T_{\mathbf{v}}(e_p) = T_{\mathbf{v}_p}(e_p) = (\varphi_p(e_p))^{-1} \lim_{N \rightarrow \infty} T_{\mathbf{v}_p}^{N+1}(e_p) = e_p.$$

From  $\lim_{N \rightarrow \infty} T_v^N(\cdot) = \varphi_{v,p}^{(r)}(\cdot)e_p$ ,  $r_{T_v} = 1$ , and 1 is a non-degenerate eigenvalue of  $T_v$ . This equality also implies  $P_{\{1\}}^{T_v} = \varphi_{v,p}^{(r)}(\cdot)e_p$ . The rest of the spectrum of  $T_v$  is in a disk  $\mathcal{B}_{s_{v,p}^{(r)}}(0)$ , for some  $0 < s_{v,p}^{(r)} < 1$ . Furthermore, we have  $\varphi_{v,p}^{(r)}(e_p)e_p = \lim T_v^N(e_p) = e_p$ . Hence  $\varphi_{v,p}^{(r)}(e_p) = 1$ . (Note that  $e_p \neq 0$  because  $\varphi_p(e_p) \neq 0$ .) By this, we have  $\varphi_{v,p}^{(r)}(1) > 0$  and  $\varphi_{v,p}^{(r)}(1)e_p = P_{\{1\}}^{T_v}(1)$ . Hence  $T_v$  satisfies the Spectral Property II with respect to  $(s_{v,p}^{(r)}, \varphi_{v,p}^{(r)}(1)e_p, \left(\varphi_{v,p}^{(r)}(1)\right)^{-1} \varphi_{v,p}^{(r)})$ .

To prove the last statement, assume that for any  $\eta \in \mathbb{C}^k$  with  $p\eta \neq 0$ , there exists an  $l_\eta \in \mathbb{N}$  such that  $(\mathcal{K}_{l_\eta}(\mathbf{v}))^* \eta = \mathbb{C}^k$ .

Let  $\rho \in \mathbf{M}_{k+}$  be the density matrix of  $\varphi_{v,p}^{(r)}$ , i.e.,  $\varphi_{v,p}^{(r)} = \text{Tr } \rho(\cdot)$ . By the  $T_v$ -invariance of  $\varphi_{v,p}^{(r)}$ , we have

$$\sum_{\mu=1}^n v_\mu^* \rho v_\mu = \rho. \quad (27)$$

We claim that for any  $\xi \in \overline{s(\rho)}\mathbb{C}^k$ , and  $\eta \in s(\rho)\mathbb{C}^k$ , we have  $\langle \eta, \mathcal{K}_l(\mathbf{v})\xi \rangle = 0$  for all  $l \in \mathbb{N}$ . As  $\eta \in s(\rho)\mathbb{C}^k$ , there exists  $c_\eta > 0$  such that  $\rho \geq c_\eta |\eta\rangle\langle \eta|$ . By the repeated use of (27), for all  $l \in \mathbb{N}$  we have

$$0 = \langle \xi, \rho \xi \rangle = \sum_{\mu^{(l)} \in \{1, \dots, n\} \times l} \left\langle \xi, (\widehat{v_{\mu^{(l)}}})^* \rho \widehat{v_{\mu^{(l)}}} \xi \right\rangle \geq c_\eta \sum_{\mu^{(l)} \in \{1, \dots, n\} \times l} |\langle \eta, \widehat{v_{\mu^{(l)}}} \xi \rangle|^2.$$

Therefore we have  $\langle \eta, \widehat{v_{\mu^{(l)}}} \xi \rangle = 0$  for any  $\mu^{(l)}$  and this proves the claim.

Next, there exists an  $\eta \in s(\rho)\mathbb{C}^k$  such that  $p\eta \neq 0$ . To see this, note that  $\varphi_{v,p}^{(r)}(p) \neq 0$ , for  $\varphi_{v,p}^{(r)}(e_p) \neq 0$ . If any  $\eta \in s(\rho)\mathbb{C}^k$  satisfies  $p\eta = 0$ , then  $s(\rho) \leq \bar{p}$  and we have  $\varphi_{v,p}^{(r)}(p) = 0$  which is a contradiction.

Let us fix some  $\eta \in s(\rho)\mathbb{C}^k$  with  $p\eta \neq 0$ . By the assumption, there exists an  $l_\eta \in \mathbb{N}$  such that  $(\mathcal{K}_{l_\eta}(\mathbf{v}))^* \eta = \mathbb{C}^k$ . Then for any  $\xi \in \overline{s(\rho)}\mathbb{C}^k$ , we have

$$\langle \mathbb{C}^k, \xi \rangle = \left\langle (\mathcal{K}_{l_\eta}(\mathbf{v}))^* \eta, \xi \right\rangle = \langle \eta, \mathcal{K}_{l_\eta}(\mathbf{v})\xi \rangle = 0,$$

by the claim. This means  $\xi = 0$ . Hence we have  $s(\rho) = 1$ , i.e.,  $\varphi_{v,p}^{(r)}$  is faithful.  $\square$

By taking the adjoint of the previous Lemma, we obtain the following Lemma.

**Lemma 2.13.** *Let  $n, k \in \mathbb{N}$ ,  $q \in \mathcal{P}(\mathbf{M}_k)$  and  $\mathbf{v} \in \mathbf{M}_k^{\times n}$ , and suppose that (iii),(v) of Condition 2 is satisfied for this  $(n, k, q, \mathbf{v})$ . Furthermore, assume that there exist constants  $c_q > 0$ ,  $0 < s_q < 1$ , a positive linear functional  $\varphi_q$  on  $\mathbf{M}_k$  and a positive element  $e_q \in \mathbf{M}_k$  such that  $s(\varphi_q), s(e_q) \leq q$ ,  $\varphi_q(e_q) \neq 0$  and*

$$\left\| T_{\mathbf{v}_q}^N(A) - \varphi_q(A)e_q \right\| \leq c_q s_q^N \|A\|, \quad \text{for all } A \in \mathbf{M}_k, \text{ and } N \in \mathbb{N}. \quad (28)$$

*Then, there exist a positive element  $e_{v,q}^{(l)} \in \mathbf{M}_k$  and a constant  $0 < s_{v,q}^{(l)} < 1$  such that  $\varphi_q(e_{v,q}^{(l)}) = 1$ , and  $T_v$  satisfies the Spectral Property II with respect to  $(s_{v,q}^{(l)}, \varphi_q(1)e_{v,q}^{(l)}, (\varphi_q(1))^{-1} \varphi_q)$ . If furthermore for any  $\eta \in \mathbb{C}^k$  with  $q\eta \neq 0$ , there exists an  $l_\eta \in \mathbb{N}$  such that  $(\mathcal{K}_{l_\eta}(\mathbf{v}))\eta = \mathbb{C}^k$ . then  $e_{v,q}^{(l)}$  is strictly positive.*

Now we are ready to prove Lemma 2.9.

**Proof of Lemma 2.9.** Let  $k_R := \text{rank } p$  and  $k_L := \text{rank } q$ . Note that  $q\mathbf{M}_k q \simeq \mathbf{M}_{k_L}$ . Under this identification, we apply Lemma 2.12 to  $(n, k_L, pq, \mathbf{v}_q)$ . The first condition of Lemma 2.12 (ii) of

*Condition 2*) can be checked by (i),(ii) of *Condition 2*. The second condition ((vi) of *Condition 2*) flows from

$$\left\| T_{\mathbf{v}(q-pq)}^N(A) \right\| = \left\| qT_{\mathbf{v}_p}^N(A)q \right\| \leq c_{\bar{p}} s_{\bar{p}}^N \|A\|, \quad A \in M_k, \quad N \in \mathbb{N}.$$

Here we used (i) (iii) of *Condition 2* for the first equality and (vi) for the last inequality. The third condition of Lemma 2.12 (i.e., (25)) is now (iv) of *Condition 2* itself. Hence, Lemma 2.12 is applicable to  $(n, k_L, pq, \mathbf{v}_q)$  and we obtain a nonzero positive linear functional  $\varphi_{\mathbf{v}_q, pq}^{(r)}$  on  $qM_k q$  and  $0 < s_{\mathbf{v}_q, pq}^{(r)} < 1$ . With respect to  $\left( s_{\mathbf{v}_q, pq}^{(r)}, \varphi_{\mathbf{v}_q, pq}^{(r)}(q)e_{pq}, \left( \varphi_{\mathbf{v}_q, pq}^{(r)}(q) \right)^{-1} \varphi_{\mathbf{v}_q, pq}^{(r)} \right)$ ,  $T_{\mathbf{v}_q}$  satisfies the *Spectral Property II*. Define a positive linear functional  $\tilde{\varphi}_{\mathbf{v}_q, pq}^{(r)}$  by  $\tilde{\varphi}_{\mathbf{v}_q, pq}^{(r)}(A) = \varphi_{\mathbf{v}_q, pq}^{(r)}(qAq)$  on  $M_k$ . Set  $\varphi_{\mathbf{v}} := \left( \tilde{\varphi}_{\mathbf{v}_q, pq}^{(r)}(1) \right)^{-1} \tilde{\varphi}_{\mathbf{v}_q, pq}^{(r)}$ . Note that  $\varphi_{\mathbf{v}}(1) = 1$ . From (vii) of *Condition 2*, we have  $s(\tilde{\varphi}_{\mathbf{v}_q, pq}^{(r)}) = q$ , by the last statement of Lemma 2.12.

Next we apply Lemma 2.13 to  $(n, k, q, \mathbf{v})$ . The first condition of Lemma 2.13 ((iii) of *Condition 2*) is (iii) of *Condition 2*. The second condition ((v) of *Condition 2*) is (v) of *Condition 2*. The third condition (28) follows from (6) of the *Spectral Property II* of  $T_{\mathbf{v}_q}$ .

Hence, Lemma 2.13 is applicable and we obtain a positive element  $e_{\mathbf{v}, q}^{(l)}$  in  $M_k$ , and  $0 < s_{\mathbf{v}, q}^{(l)} < 1$ . Set  $e_{\mathbf{v}} = e_{\mathbf{v}, q}^{(l)}$  and  $s_{\mathbf{v}} = s_{\mathbf{v}, q}^{(l)}$ . With respect to  $(s_{\mathbf{v}}, e_{\mathbf{v}}, \varphi_{\mathbf{v}})$ ,  $T_{\mathbf{v}}$  satisfies the *Spectral Property II*.

We know  $s(\varphi_{\mathbf{v}}) = s(\tilde{\varphi}_{\mathbf{v}_q, pq}^{(r)}) = q$ . We show  $s(e_{\mathbf{v}}) = p$  in the rest of the proof.

To see this, we apply Lemma 2.13 to  $(n, k_R, pq, \mathbf{v}_p)$ , under the identification  $pM_k p \simeq M_{k_R}$ .

The first condition of Lemma 2.13 ((iii) of *Condition 2*) can be checked by (i),(iii) of *Condition 2*. The second condition ((v) of *Condition 2*) follows from

$$\left\| T_{\mathbf{v}_p - pq}^N(A) \right\| = \left\| T_{\mathbf{v}_q}^N(pAp) \right\| \leq c_{\bar{q}} s_{\bar{q}}^N \|A\|, \quad A \in M_k, \quad N \in \mathbb{N}.$$

Here we used (i) (ii) of *Condition 2* for the first equality and (v) for the last inequality. The third condition (28) is (iv) of *Condition 2* itself.

Hence, Lemma 2.13 is applicable and we obtain a positive element  $e_{\mathbf{v}_p, pq}^{(l)}$  in  $pM_k p$ , and  $0 < s_{\mathbf{v}_p, pq}^{(l)} < 1$ . With respect to  $\left( s_{\mathbf{v}_p, pq}^{(l)}, \varphi_{pq}(1)e_{\mathbf{v}_p, pq}^{(l)}, \varphi_{pq}(1)^{-1} \varphi_{pq} \right)$ ,  $T_{\mathbf{v}_p}$  satisfies the *Spectral Property II*. By (viii) of *Condition 2*,  $s(e_{\mathbf{v}_p, pq}^{(l)}) = p$ .

As  $\varphi_{\mathbf{v}}$ ,  $\varphi_{pq}$  are faithful on  $pqM_k pq$ , we have  $\varphi_{\mathbf{v}}(pq), \varphi_{pq}(pq) > 0$ . By

$$\varphi_{\mathbf{v}}(pq)e_{\mathbf{v}} = \lim_{N \rightarrow \infty} T_{\mathbf{v}}^N(pq) = \lim_{N \rightarrow \infty} T_{\mathbf{v}_p}^N(pq) = \varphi_{pq}(pq)e_{\mathbf{v}_p, pq}^{(l)},$$

we obtain  $s(e_{\mathbf{v}}) = s(e_{\mathbf{v}_p, pq}^{(l)}) = p$ . □

## 2.2 The intersection property

In this subsection we prove the following Lemma.

**Lemma 2.14.** *Let  $m_2, m_3 \in \mathbb{N}$ . Let  $n, k \in \mathbb{N}$  and  $\mathbf{v} \in M_k^{\times n}$ . Assume that the triple  $(n, k, \mathbf{v})$  satisfies the Condition 4 for  $(m_2, m_3)$  with respect to  $X_{m_2} \in \mathcal{K}_{m_2}(\mathbf{v})$ . Then,*

(i) *for all  $N \geq m_3$ ,*

$$(X_{m_2}^{-1})^* \ker \Gamma_{N, \mathbf{v}}^{(R)} \subset \ker \Gamma_{N+m_2, \mathbf{v}}^{(R)}, \text{ and}$$

(ii) *for all  $N \geq m_2 + m_3 + 1$ ,*

$$\mathcal{G}_{N, \mathbf{v}} = (\mathcal{G}_{N-1, \mathbf{v}} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{N-1, \mathbf{v}}).$$

**Proof.** First we prove (i). For any  $N \geq m_3$  and  $Y \in \ker \Gamma_{N,\mathbf{v}}^{(R)}$ , by the definition of  $\Gamma_{N,\mathbf{v}}^{(R)}$ , we have  $\text{Tr } YZ^* = 0$  for any  $Z \in \mathcal{K}_N(\mathbf{v})$ . By the assumption, for any  $\mu^{(N+m_2)} \in \{1, \dots, n\}^{\times(N+m_2)}$ , we have  $X_{m_2}^{-1} \widehat{v_{\mu^{(N+m_2)}}} \in \mathcal{K}_N(\mathbf{v})$ . From these two observations, we have

$$\begin{aligned} \Gamma_{N+m_2,\mathbf{v}}^{(R)}((X_{m_2}^{-1})^* Y) &= \sum_{\mu^{(N+m_2)} \in \{1, \dots, n\}^{\times(N+m_2)}} \left( \text{Tr} \left( (X_{m_2}^{-1})^* Y \left( \widehat{v_{\mu^{(N+m_2)}}} \right)^* \right) \right) \widehat{\psi_{\mu^{(N+m_2)}}} \\ &= \sum_{\mu^{(N+m_2)} \in \{1, \dots, n\}^{\times(N+m_2)}} \left( \text{Tr} \left( Y \left( (X_{m_2}^{-1} \widehat{v_{\mu^{(N+m_2)}}})^* \right) \right) \right) \widehat{\psi_{\mu^{(N+m_2)}}} = 0. \end{aligned}$$

Hence we have proven (i).

The inclusion  $\mathcal{G}_{N,\mathbf{v}} \subset (\mathcal{G}_{N-1,\mathbf{v}} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{N-1,\mathbf{v}})$  for any  $2 \leq N \in \mathbb{N}$  can be seen by

$$\begin{aligned} \Gamma_{N,\mathbf{v}}^{(R)}(Y) &= \sum_{\nu \in \{1, \dots, n\}} \sum_{\mu^{(N-1)} \in \{1, \dots, n\}^{\times(N-1)}} \left( \text{Tr} \left( Y v_\nu^* \left( \widehat{v_{\mu^{(N-1)}}} \right)^* \right) \right) \widehat{\psi_{\mu^{(N-1)}}} \otimes \psi_\nu \\ &= \sum_{\nu \in \{1, \dots, n\}} \Gamma_{N-1,\mathbf{v}}^{(R)}(Y v_\nu^*) \otimes \psi_\nu \\ &= \sum_{\nu \in \{1, \dots, n\}} \sum_{\mu^{(N-1)} \in \{1, \dots, n\}^{\times(N-1)}} \left( \text{Tr} \left( Y \left( \widehat{v_{\mu^{(N-1)}}} \right)^* v_\nu^* \right) \right) \psi_\nu \otimes \widehat{\psi_{\mu^{(N-1)}}} \\ &= \sum_{\nu \in \{1, \dots, n\}} \psi_\nu \otimes \Gamma_{N-1,\mathbf{v}}^{(R)}(v_\nu^* Y). \end{aligned}$$

To see the opposite inclusion, let  $N \geq m_2 + m_3 + 1$  and  $\Phi \in (\mathcal{G}_{N-1,\mathbf{v}} \otimes \mathbb{C}^n) \cap (\mathbb{C}^n \otimes \mathcal{G}_{N-1,\mathbf{v}})$ . Then by some sets of  $k \times k$  matrices,  $\{C_\mu\}_{\mu=1}^n$  and  $\{D_\nu\}_{\nu=1}^n$ , we can write  $\Phi$  as

$$\Phi = \sum_{\mu=1}^n \psi_\mu \otimes \Gamma_{N-1,\mathbf{v}}^{(R)}(C_\mu) = \sum_{\nu=1}^n \Gamma_{N-1,\mathbf{v}}^{(R)}(D_\nu) \otimes \psi_\nu. \quad (29)$$

From this relation and the definition of  $\Gamma_{N,\mathbf{v}}^{(R)}$ , we have for all  $\mu, \nu, \mu_j \in \{1, \dots, n\}$ ,  $2 \leq j \leq m_2$  and  $\mu_i \in \{1, \dots, n\}$ ,  $m_2 + 1 \leq i \leq N - 1$ ,

$$\text{Tr} \left( \left( v_{\mu_{m_2}}^* \cdots v_{\mu_2}^* C_\mu v_\nu^* - v_{\mu_{m_2}}^* \cdots v_{\mu_2}^* v_\mu^* D_\nu \right) v_{\mu_{N-1}}^* \cdots v_{\mu_{m_2+1}}^* \right) = 0.$$

Therefore, for all  $\mu, \nu, \mu_j \in \{1, \dots, n\}$ ,  $2 \leq j \leq m_2$ ,

$$v_{\mu_{m_2}}^* \cdots v_{\mu_2}^* C_\mu v_\nu^* - v_{\mu_{m_2}}^* \cdots v_{\mu_2}^* v_\mu^* D_\nu \in \ker \Gamma_{N-1-m_2,\mathbf{v}}^{(R)}. \quad (30)$$

(If  $m_2 = 1$ , replace  $v_{\mu_{m_2}}^* \cdots v_{\mu_2}^*$  by 1, and understand (30) as  $C_\mu v_\nu^* - v_\mu^* D_\nu \in \ker \Gamma_{N-2,\mathbf{v}}^{(R)}$ .)

We claim that there exists  $\tilde{C} \in \text{M}_k$  such that  $\tilde{C} v_\nu^* - D_\nu \in \ker \Gamma_{N-1,\mathbf{v}}^{(R)}$  for all  $\nu \in \{1, \dots, n\}$ . As  $X_{m_2} \in \mathcal{K}_{m_2}(\mathbf{v})$ , there exists a set of coefficients  $\{\alpha_{\mu^{(m_2)}}\}_{\mu^{(m_2)} \in \{1, \dots, n\}^{\times m_2}} \subset \mathbb{C}$  such that  $X_{m_2} = \sum_{\mu^{(m_2)}} \alpha_{\mu^{(m_2)}} \widehat{v_{\mu^{(m_2)}}}$ . On the other hand, as  $N - 1 - m_2 \geq m_3$ , (30) implies

$$(X_{m_2}^{-1})^* \left( v_{\mu_{m_2}}^* \cdots v_{\mu_2}^* C_{\mu_1} v_\nu^* - v_{\mu_{m_2}}^* \cdots v_{\mu_2}^* v_{\mu_1}^* D_\nu \right) \in (X_{m_2}^{-1})^* \ker \Gamma_{N-1-m_2,\mathbf{v}}^{(R)} \subset \ker \Gamma_{N-1,\mathbf{v}}^{(R)}, \quad (31)$$

for all  $\nu, \mu_j \in \{1, \dots, n\}$ ,  $1 \leq j \leq m_2$ . We used (i) for the last inclusion. Set

$$\tilde{C} := \sum_{\mu^{(m_2)} = (\mu_1, \dots, \mu_{m_2})} \overline{\alpha_{\mu^{(m_2)}}} (X_{m_2}^{-1})^* v_{\mu_{m_2}}^* \cdots v_{\mu_2}^* C_{\mu_1} \in \text{M}_k.$$

From (31), we obtain

$$\begin{aligned} \tilde{C}v_\nu^* - D_\nu &= \tilde{C}v_\nu^* - (X_{m_2}^{-1})^* X_{m_2}^* D_\nu \\ &= \sum_{\mu^{(m_2)}=(\mu_1, \dots, \mu_{m_2})} \overline{\alpha_{\mu^{(m_2)}}} (X_{m_2}^{-1})^* \left( v_{\mu_{m_2}}^* \cdots v_{\mu_2}^* C_{\mu_1} v_\nu^* - v_{\mu_{m_2}}^* \cdots v_{\mu_2}^* v_{\mu_1}^* D_\nu \right) \in \ker \Gamma_{N-1, \mathbf{v}}^{(R)}, \end{aligned}$$

for all  $\nu \in \{1, \dots, n\}$ , proving the claim.

Substituting this to (29), we have

$$\Phi = \sum_{\mu_N=1}^n \Gamma_{N-1, \mathbf{v}}^{(R)}(D_{\mu_N}) \otimes \psi_{\mu_N} = \sum_{\mu_N=1}^n \Gamma_{N-1, \mathbf{v}}^{(R)}(\tilde{C}v_{\mu_N}^*) \otimes \psi_{\mu_N} = \Gamma_{N, \mathbf{v}}^{(R)}(\tilde{C}) \in \mathcal{G}_{N, \mathbf{v}}.$$

□

### 2.3 Bijectivity of $\Gamma_{N, \mathbf{v}}^{(R)}|_{p\mathbf{M}_k q}$

In this section, we prove the bijectivity of  $\Gamma_{N, \mathbf{v}}^{(R)}$  on  $p\mathbf{M}_k q$  for  $N$  large enough, under *Condition 2* and *Condition 3*. Recall the notations (21) and (8)

**Lemma 2.15.** *Let  $n, k \in \mathbb{N}$ ,  $p, q \in \mathcal{P}(\mathbf{M}_k)$ , and  $\mathbf{v} \in \mathbf{M}_k^{\times n}$ . Suppose that the pentad  $(n, k, p, q, \mathbf{v})$  satisfies the Condition 2, and the Condition 3 for some  $m_1 \in \mathbb{N}$ . Then*

1.  $M_{\mathbf{v}, p, q} \leq L_{\mathbf{v}} < \infty$ , i.e.,  $\Gamma_{N, \mathbf{v}}^{(R)}$  is injective on  $p\mathbf{M}_k q$  for large  $N$ ,
2. for all  $N \geq \max\{M_{\mathbf{v}, p, q}, m_1\}$ , the map  $\Gamma_{N, \mathbf{v}}^{(R)}|_{p\mathbf{M}_k q} : p\mathbf{M}_k q \rightarrow \mathcal{G}_{N, \mathbf{v}}$  is bijective.

We first introduce the Lemma which we will use repeatedly.

**Lemma 2.16.** *Let  $n, k \in \mathbb{N}$ ,  $p, q \in \mathcal{P}(\mathbf{M}_k)$ , and  $\mathbf{v} \in \mathbf{M}_k^{\times n}$  satisfying the Condition 2. Then for any  $X, Y \in p\mathbf{M}_k q$  and  $N \in \mathbb{N}$ , we have*

$$\left| \left\langle \Gamma_{N, \mathbf{v}}^{(R)}(X), \Gamma_{N, \mathbf{v}}^{(R)}(Y) \right\rangle - \langle X, Y \rangle_{\mathbf{v}} \right| \leq E_{\mathbf{v}}(N) \langle X, X \rangle_{\mathbf{v}}^{\frac{1}{2}} \langle Y, Y \rangle_{\mathbf{v}}^{\frac{1}{2}}, \quad (32)$$

and

$$(1 - E_{\mathbf{v}}(N)) \langle X, X \rangle_{\mathbf{v}} \leq \left\| \Gamma_{N, \mathbf{v}}^{(R)}(X) \right\|^2 \leq (1 + E_{\mathbf{v}}(N)) \langle X, X \rangle_{\mathbf{v}}. \quad (33)$$

Furthermore, for  $X \in p\mathbf{M}_k q$  and  $N \in \mathbb{N}$  with  $N \geq L_{\mathbf{v}}$ , we have

$$\|X\|_2 \leq \sqrt{\frac{2}{a_{\mathbf{v}} c_{\mathbf{v}}}} \left\| \Gamma_{N, \mathbf{v}}^{(R)}(X) \right\|, \quad (34)$$

and  $\Gamma_{N, \mathbf{v}}^{(R)}$  is injective on  $p\mathbf{M}_k q$ .

**Proof.** The proof is basically Section 5 of [FNW], but the fact that our  $e_{\mathbf{v}}, \rho_{\mathbf{v}}$  are not strictly positive requires an additional argument. As in Lemma 5.2 of [FNW], we have for any  $X, Y \in p\mathbf{M}_k q$  and  $N \in \mathbb{N}$ ,

$$\begin{aligned} &\left\langle \Gamma_{N, \mathbf{v}}^{(R)}(X), \Gamma_{N, \mathbf{v}}^{(R)}(Y) \right\rangle \\ &= \langle X, Y \rangle_{\mathbf{v}} + \sum_{i=1}^k \sum_{j=1}^k \left\langle \chi_i^{(k)}, T_{\mathbf{v}}^N \circ \left( \mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}}} \right) \left( X^* \left| \chi_i^{(k)} \right\rangle \left\langle \chi_j^{(k)} \right| Y \right) \chi_j^{(k)} \right\rangle, \end{aligned} \quad (35)$$



and

$$\begin{aligned} \left| \left\langle \Gamma_{N,\mathbf{v}}^{(R)}(X), \Gamma_{N,\mathbf{v}}^{(R)}(Y) \right\rangle - \langle X, Y \rangle_{\mathbf{v}} \right| &\leq k^2 \left\| T_{\mathbf{v}}^N \left( 1 - P_{\{1\}}^{T_{\mathbf{v}}} \right) \right\| \|X\| \|Y\| \\ &= \tilde{E}_{\mathbf{v}}(N) \|X\| \|Y\| \leq E_{\mathbf{v}}(N) \langle X, X \rangle_{\mathbf{v}}^{\frac{1}{2}} \langle Y, Y \rangle_{\mathbf{v}}^{\frac{1}{2}}. \end{aligned}$$

Here we used (23) for the last inequality. The inequality (33) is clear from (32). Furthermore, if  $N \geq L_{\mathbf{v}}$ , we have  $E_{\mathbf{v}}(N) < \frac{1}{2}$ . Substituting this to (33) and using (23), we obtain (34). The injectivity is clear from this inequality.  $\square$

Now we prove Lemma 2.15

**Proof of Lemma 2.15.** 1. was already proven in Lemma 2.16. To see 2, let  $N \geq \max\{M_{\mathbf{v},p,q}, m_1\}$ . As  $\Gamma_{N,\mathbf{v}}^{(R)}$  is injective on  $pM_kq$ , we have

$$(\text{rank } p)(\text{rank } q) = \dim pM_kq = \dim \Gamma_{N,\mathbf{v}}^{(R)}(pM_kq) \leq \dim \Gamma_{N,\mathbf{v}}^{(R)}(M_k) = (\text{rank } p)(\text{rank } q),$$

and  $\Gamma_{N,\mathbf{v}}^{(R)}(pM_kq) \subset \Gamma_{N,\mathbf{v}}^{(R)}(M_k)$ . Hence we have  $\Gamma_{N,\mathbf{v}}^{(R)}(pM_kq) = \Gamma_{N,\mathbf{v}}^{(R)}(M_k)$ , proving 2.  $\square$

## 2.4 Estimation of the overlaps of projections

In this section we prove the following Lemma.

**Lemma 2.17.** *Let  $n, k \in \mathbb{N}$ ,  $p, q \in \mathcal{P}(M_k)$ , and  $\mathbf{v} \in M_k^{\times n}$ . Suppose that the pentad  $(n, k, p, q, \mathbf{v})$  satisfies the Condition 2 and the Condition 3 for some  $m_1 \in \mathbb{N}$ . Then for all  $l, m, r \in \mathbb{N}$  with  $m \geq \max\{m_1, L_{\mathbf{v}}\}$ , we have*

$$\left\| (\mathbb{I}_{[0,l-1]} \otimes G_{m+r,\mathbf{v}}) (G_{l+m,\mathbf{v}} \otimes \mathbb{I}_{[l+m,l+m+r-1]} - G_{l+m+r,\mathbf{v}}) \right\| \leq 2F_{\mathbf{v}}E_{\mathbf{v}}(m) (F_{\mathbf{v}}^2E_{\mathbf{v}}(m) + 1). \quad (36)$$

The proof goes parallel to Lemma 6.2 of [FNW]. However, again the fact that our  $e_{\mathbf{v}}, \rho_{\mathbf{v}}$  are not strictly positive requires additional arguments.

**Lemma 2.18.** *Let  $n, k \in \mathbb{N}$ ,  $p, q \in \mathcal{P}(M_k)$ , and  $\mathbf{v} \in M_k^{\times n}$  satisfying Condition 2. Let  $l, m, r \in \mathbb{N}$ ,  $\Phi \in \mathcal{G}_{l+m,\mathbf{v}} \otimes (\mathbb{C}^n)^{\otimes r}$  and  $\Psi \in (\mathbb{C}^n)^{\otimes l} \otimes \mathcal{G}_{m+r,\mathbf{v}}$ . Let  $\mathfrak{P}^{\Phi} := \{\tilde{\Phi}_{\mu^{(r)}}\}_{\mu^{(r)} \in \{1, \dots, n\}^{\times r}} \subset M_k$ ,  $\mathfrak{P}_{\Psi} := \{\tilde{\Psi}_{\mu^{(l)}}\}_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \subset M_k$  be sets of matrices such that*

$$\Phi = \sum_{\mu^{(r)}} \Gamma_{l+m,\mathbf{v}}^{(R)} \left( \tilde{\Phi}_{\mu^{(r)}} \right) \otimes \widehat{\psi_{\mu^{(r)}}}, \quad \Psi = \sum_{\mu^{(l)}} \widehat{\psi_{\mu^{(l)}}} \otimes \Gamma_{m+r,\mathbf{v}}^{(R)} \left( \tilde{\Psi}_{\mu^{(l)}} \right).$$

Define

$$V^{\mathfrak{P}^{\Phi}} := \sum_{\mu^{(r)}} \tilde{\Phi}_{\mu^{(r)}} \rho_{\mathbf{v}} \widehat{v_{\mu^{(r)}}} y_{\mathbf{v}}, \quad V_{\mathfrak{P}_{\Psi}} := \sum_{\mu^{(l)}} x_{\mathbf{v}} \widehat{v_{\mu^{(l)}}} e_{\mathbf{v}} \tilde{\Psi}_{\mu^{(l)}} \in M_k. \quad (37)$$

Then we have

$$\left| \langle \Phi, \Psi \rangle - \left\langle V^{\mathfrak{P}^{\Phi}}, V_{\mathfrak{P}_{\Psi}} \right\rangle_{\mathbf{v}} \right| \leq k \left\| T_{\mathbf{v}}^m \left( \mathbb{I} - P_{\{1\}}^{T_{\mathbf{v}}} \right) \right\| \left( \text{Tr } T_{\mathbf{v}}^l(1) \sum_{\mu^{(r)}} \tilde{\Phi}_{\mu^{(r)}} \tilde{\Phi}_{\mu^{(r)}}^* \right)^{\frac{1}{2}} \left( \text{Tr } T_{\mathbf{v}}^r \left( \sum_{\mu^{(l)}} \tilde{\Psi}_{\mu^{(l)}}^* \tilde{\Psi}_{\mu^{(l)}} \right) \right)^{\frac{1}{2}} \quad (38)$$

*Remark 2.19.* Note that such  $\mathfrak{P}^{\Phi}, \mathfrak{P}_{\Psi}$  are not necessarily unique.

**Proof.** As in the proof of Lemma 2.16, it is straightforward to show that

$$\langle \Phi, \Psi \rangle - \sum_{\mu^{(r)} \mu^{(l)}} \varphi_{\mathbf{v}} \left( \tilde{\Phi}_{\mu^{(r)}}^* \widehat{v_{\mu^{(l)}}} e_{\mathbf{v}} \tilde{\Psi}_{\mu^{(l)}} \left( \widehat{v_{\mu^{(r)}}} \right)^* \right) \quad (39)$$

is bounded by the right hand side of (38). We rewrite the second term. Note that  $(\widehat{v_{\mu^{(l)}}}) p = p (\widehat{v_{\mu^{(l)}}}) p$  and  $(\widehat{v_{\mu^{(r)}}})^* q = q (\widehat{v_{\mu^{(r)}}})^* q$ . From this and the relations  $e_{\mathbf{v}} x_{\mathbf{v}} = p$ ,  $\rho_{\mathbf{v}} y_{\mathbf{v}} = q$ ,  $s(e_{\mathbf{v}}) = p$  and  $s(\rho_{\mathbf{v}}) = q$ , we have

$$\begin{aligned} \sum_{\mu^{(r)} \mu^{(l)}} \varphi_{\mathbf{v}} \left( \tilde{\Phi}_{\mu^{(r)}}^* \widehat{v_{\mu^{(l)}}} e_{\mathbf{v}} \tilde{\Psi}_{\mu^{(l)}} \left( \widehat{v_{\mu^{(r)}}} \right)^* \right) &= \sum_{\mu^{(r)} \mu^{(l)}} \text{Tr} \left( \rho_{\mathbf{v}} y_{\mathbf{v}} \left( \widehat{v_{\mu^{(r)}}} \right)^* \rho_{\mathbf{v}} \tilde{\Phi}_{\mu^{(r)}}^* e_{\mathbf{v}} x_{\mathbf{v}} \widehat{v_{\mu^{(l)}}} e_{\mathbf{v}} \tilde{\Psi}_{\mu^{(l)}} \right) \\ &= \varphi_{\mathbf{v}} \left( \left( V^{\mathfrak{P}^{\Phi}} \right)^* e_{\mathbf{v}} (V_{\mathfrak{P}^{\Psi}}) \right) = \left\langle V^{\mathfrak{P}^{\Phi}}, V_{\mathfrak{P}^{\Psi}} \right\rangle_{\mathbf{v}}. \end{aligned}$$

□

**Lemma 2.20.** Let  $n, k \in \mathbb{N}$ ,  $p, q \in \mathcal{P}(\mathbb{M}_k)$ , and  $\mathbf{v} \in \mathbb{M}_k^{\times n}$ . Suppose that the pentad  $(n, k, p, q, \mathbf{v})$  satisfies the Condition 2 and the Condition 3 for some  $m_1 \in \mathbb{N}$ . Let  $l, m, r \in \mathbb{N}$  with  $m \geq \max\{m_1, L_{\mathbf{v}}\}$ ,  $\Phi \in \mathcal{G}_{l+m, \mathbf{v}} \otimes (\mathbb{C}^n)^{\otimes r}$  and  $\Psi \in (\mathbb{C}^n)^{\otimes l} \otimes \mathcal{G}_{m+r, \mathbf{v}}$ . Then

(1) there exist unique  $\Omega^{\Phi} := \{\Phi_{\mu^{(r)}}\}_{\mu^{(r)} \in \{1, \dots, n\} \times r} \subset p\mathbb{M}_k q$ ,  $\Omega_{\Psi} := \{\Psi_{\mu^{(l)}}\}_{\mu^{(l)} \in \{1, \dots, n\} \times l} \subset p\mathbb{M}_k q$  such that

$$\Phi = \sum_{\mu^{(r)}} \Gamma_{l+m, \mathbf{v}}^{(R)} (\Phi_{\mu^{(r)}}) \otimes \widehat{\psi_{\mu^{(r)}}}, \quad \Psi = \sum_{\mu^{(l)}} \widehat{\psi_{\mu^{(l)}}} \otimes \Gamma_{m+r, \mathbf{v}}^{(R)} (\Psi_{\mu^{(l)}}).$$

(2) For  $\Omega^{\Phi}$ ,  $\Omega_{\Psi}$  of (1), we have

$$\text{Tr} \left( T_{\mathbf{v}}^l(1) \left( \sum_{\mu^{(r)}} \Phi_{\mu^{(r)}} \Phi_{\mu^{(r)}}^* \right) \right) \leq \frac{2F_{\mathbf{v}}}{a_{\mathbf{v}} c_{\mathbf{v}}} \|\Phi\|^2, \quad \text{Tr} \left( T_{\mathbf{v}}^r \left( \sum_{\mu^{(l)}} \Psi_{\mu^{(l)}}^* \Psi_{\mu^{(l)}} \right) \right) \leq \frac{2kF_{\mathbf{v}}}{a_{\mathbf{v}} c_{\mathbf{v}}} \|\Psi\|^2.$$

(3) For any  $X \in p\mathbb{M}_k q$ , we have

$$\begin{aligned} \text{Tr} \left( T_{\mathbf{v}}^l(1) \sum_{\mu^{(r)}} \left( X \left( \widehat{v_{\mu^{(r)}}} \right)^* \right) \left( \left( X \left( \widehat{v_{\mu^{(r)}}} \right)^* \right)^* \right) \right) &\leq \frac{kF_{\mathbf{v}}^2}{a_{\mathbf{v}} c_{\mathbf{v}}} \langle X, X \rangle_{\mathbf{v}} \\ \text{Tr} \left( T_{\mathbf{v}}^r \left( \sum_{\mu^{(l)}} \left( \left( \widehat{v_{\mu^{(l)}}} \right)^* X \right)^* \left( \left( \widehat{v_{\mu^{(l)}}} \right)^* X \right) \right) \right) &\leq \frac{kF_{\mathbf{v}}^2}{a_{\mathbf{v}} c_{\mathbf{v}}} \langle X, X \rangle_{\mathbf{v}} \end{aligned}$$

**Proof.** (1) follows from Lemma 2.15. The first inequality of (2) can be checked as

$$\begin{aligned} \text{Tr} \left( T_{\mathbf{v}}^l(1) \left( \sum_{\mu^{(r)}} \Phi_{\mu^{(r)}} \Phi_{\mu^{(r)}}^* \right) \right) &\leq \|T_{\mathbf{v}}^l(1)\| \text{Tr} \left( \sum_{\mu^{(r)}} \Phi_{\mu^{(r)}} \Phi_{\mu^{(r)}}^* \right) \\ &\leq F_{\mathbf{v}} \text{Tr} \left( \sum_{\mu^{(r)}} \Phi_{\mu^{(r)}} \Phi_{\mu^{(r)}}^* \right) \leq \frac{2F_{\mathbf{v}}}{a_{\mathbf{v}} c_{\mathbf{v}}} \sum_{\mu^{(r)}} \left\| \Gamma_{l+m, \mathbf{v}}^{(R)} (\Phi_{\mu^{(r)}}) \right\|^2 = \frac{2F_{\mathbf{v}}}{a_{\mathbf{v}} c_{\mathbf{v}}} \|\Phi\|^2 \end{aligned}$$

Here we used (22) for the second inequality and (34) for the third inequality with  $m \geq L_v$ . The second one can be obtained similarly. The first inequality of (3) can be seen for  $X \in pM_k q$ , by

$$\text{Tr} \left( T_v^l(1) \sum_{\mu^{(r)}} \left( X (\widehat{v_{\mu^{(r)}}})^* \right) \left( \left( X (\widehat{v_{\mu^{(r)}}})^* \right)^* \right) \right) = \text{Tr} (T_v^r (X^* T_v^l(1) X)) \leq k F_v^2 \|X\|^2 \leq \frac{k F_v^2}{a_v c_v} \langle X, X \rangle_v.$$

Here we used (22) for the first inequality and (23) for the second inequality. The second inequality of (3) can be obtained similarly.  $\square$

**Lemma 2.21.** *Let  $n, k \in \mathbb{N}$ ,  $p, q \in \mathcal{P}(M_k)$ , and  $v \in M_k^{\times n}$ . Suppose that the pentad  $(n, k, p, q, v)$  satisfies the Condition 2 and the Condition 3 for some  $m_1 \in \mathbb{N}$ . Let  $l, m, r \in \mathbb{N}$  with  $m \geq \max\{m_1, L_v\}$ ,  $\Phi \in (\mathcal{G}_{l+m, v} \otimes (\mathbb{C}^n)^{\otimes r}) \cap \mathcal{G}_{l+m+r, v}^\perp$  and  $\Psi \in ((\mathbb{C}^n)^{\otimes l} \otimes \mathcal{G}_{m+r, v}) \cap \mathcal{G}_{l+m+r, v}^\perp$ . Let  $\Omega^\Phi := \{\Phi_{\mu^{(r)}}\}_{\mu^{(r)} \in \{1, \dots, n\}^{\times r}} \subset pM_k q$ ,  $\Omega_\Psi := \{\Psi_{\mu^{(l)}}\}_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \subset pM_k q$  as in (1) of Lemma 2.20. Then, for  $V^{\Omega^\Phi}$ ,  $V_{\Omega_\Psi}$  defined as in (37), we have*

$$\left\langle V^{\Omega^\Phi}, V^{\Omega^\Phi} \right\rangle_v^{\frac{1}{2}} \leq \sqrt{2} F_v^{\frac{3}{2}} E_v(m) \|\Phi\|, \quad \langle V_{\Omega_\Psi}, V_{\Omega_\Psi} \rangle_v^{\frac{1}{2}} \leq \sqrt{2} F_v^{\frac{3}{2}} E_v(m) \|\Psi\|.$$

**Proof.** Take an arbitrary  $X \in pM_k q$ . We show

$$\left| \left\langle V^{\Omega^\Phi}, X \right\rangle_v \right| \leq \sqrt{2} F_v^{\frac{3}{2}} E_v(m) \|\Phi\| \langle X, X \rangle_v^{\frac{1}{2}}, \quad |\langle X, V_{\Omega_\Psi} \rangle_v| \leq \sqrt{2} F_v^{\frac{3}{2}} E_v(m) \|\Psi\| \langle X, X \rangle_v^{\frac{1}{2}}. \quad (40)$$

To see this, note that

$$\Gamma_{l+m+r, v}^{(R)}(X) = \sum_{\mu^{(r)}} \Gamma_{l+m, v}^{(R)} \left( X (\widehat{v_{\mu^{(r)}}})^* \right) \otimes \widehat{\psi_{\mu^{(r)}}} = \sum_{\mu^{(l)}} \widehat{\psi_{\mu^{(l)}}} \otimes \Gamma_{m+r, v}^{(R)} \left( (\widehat{v_{\mu^{(l)}}})^* X \right).$$

Set  $\mathfrak{P}_{l+m+r, v}^{\Gamma^{(R)}}(X) := \{X (\widehat{v_{\mu^{(r)}}})^*\}_{\mu^{(r)}}, \mathfrak{P}_{\Gamma_{l+m+r, v}^{(R)}}(X) := \{(\widehat{v_{\mu^{(l)}}})^* X\}_{\mu^{(l)}}$  and consider the  $V \mathfrak{P}_{l+m+r, v}^{\Gamma^{(R)}}(X)$ ,

$V_{\mathfrak{P}_{\Gamma_{l+m+r, v}^{(R)}}(X)}$  given by the formula (37). Then we find  $V \mathfrak{P}_{l+m+r, v}^{\Gamma^{(R)}}(X) = V_{\mathfrak{P}_{\Gamma_{l+m+r, v}^{(R)}}(X)} = X$ .

For  $\Phi$  and  $\Psi$ , consider the unique  $\Omega^\Phi := \{\Phi_{\mu^{(r)}}\}_{\mu^{(r)} \in \{1, \dots, n\}^{\times r}} \subset pM_k q$ ,  $\Omega_\Psi := \{\Psi_{\mu^{(l)}}\}_{\mu^{(l)} \in \{1, \dots, n\}^{\times l}} \subset pM_k q$  given in Lemma 2.20. Then, by Lemma 2.18 and Lemma 2.20 and  $\Phi \in \mathcal{G}_{l+m+r, v}^\perp$ , we have

$$\begin{aligned} \left| \left\langle V^{\Omega^\Phi}, X \right\rangle_v \right| &= \left| \left\langle V^{\Omega^\Phi}, V_{\mathfrak{P}_{\Gamma_{l+m+r, v}^{(R)}}(X)} \right\rangle_v - \left\langle \Phi, \Gamma_{l+m+r, v}^{(R)}(X) \right\rangle \right| \\ &\leq k \left\| T_v^m (\mathbb{I} - P_{\{1\}}^{T_v}) \right\| \sqrt{\frac{2 F_v}{a_v c_v}} \|\Phi\| \sqrt{\frac{k F_v^2}{a_v c_v}} \langle X, X \rangle_v^{\frac{1}{2}} \leq \sqrt{2} F_v^{\frac{3}{2}} E_v(m) \|\Phi\| \langle X, X \rangle_v^{\frac{1}{2}}, \end{aligned}$$

proving the first inequality of (40). From (24) and (40), we obtain the first inequality of Lemma 2.21. The second inequality can be seen in the same manner.  $\square$

Now we are ready to prove Lemma 2.17.

**Proof of Lemma 2.17.** In the setting of Lemma 2.17, we fix arbitrary  $\Phi \in (\mathcal{G}_{l+m, v} \otimes (\mathbb{C}^n)^{\otimes r}) \cap \mathcal{G}_{l+m+r, v}^\perp$  and  $\Psi \in ((\mathbb{C}^n)^{\otimes l} \otimes \mathcal{G}_{m+r, v}) \cap \mathcal{G}_{l+m+r, v}^\perp$ . From Lemma 2.21, we have

$$\left| \left\langle V^{\Omega^\Phi}, V_{\Omega_\Psi} \right\rangle_v \right| \leq \left\langle V^{\Omega^\Phi}, V^{\Omega^\Phi} \right\rangle_v^{\frac{1}{2}} \langle V_{\Omega_\Psi}, V_{\Omega_\Psi} \rangle_v^{\frac{1}{2}} \leq 2 F_v^3 E_v(m)^2 \|\Phi\| \|\Psi\|.$$

Combining this with Lemma 2.20 and Lemma 2.18, we obtain

$$|\langle \Phi, \Psi \rangle| \leq 2 E_v(m) F_v (E_v(m) F_v^2 + 1) \|\Phi\| \|\Psi\|.$$

This completes the proof.  $\square$

## 2.5 Proof of Proposition 2.6

**Proof of Proposition 2.6.** As  $r_{T_v} = 1$ ,  $v$  is nonzero and  $\mathcal{G}_{l,v}$  is a nonzero subspace of  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$ . That  $M_{v,p,q} < \infty$  is Lemma 2.15. Set  $m_0 = m_2 + m_3$ . Then (ii) of Lemma 2.14 implies Property (I,  $m_0$ ), i.e., (i) of *Condition 1* holds.

To prove (ii) of *Condition 1*, choose  $L_1 \in \mathbb{N}$  such that

$$4\sqrt{m}F_v E_v(m) (F_v^2 E_v(m) + 1) < 1,$$

for all  $m \geq L_1$ . This is possible because of the spectral property of  $T_v$ . Set  $l_0 := \max\{m_1, L_1, L_v\} + 1 \in \mathbb{N}$ . For any  $l \geq l_0$  and  $n \geq 2l$ , we use Lemma 2.17 replacing  $(l, m, r)$  in Lemma 2.17 by  $(n - l + 1, l - 1, 1)$ . Then we obtain (ii) for this  $l_0$ .  $\square$

## 3 Properties of the ground state structure of $H_{\Phi_{m,\mathbb{B}}}$ for $\mathbb{B} \in \text{ClassA}$

In this section we prove Theorem 1.18 of the Hamiltonian  $H_{\Phi_{m,\mathbb{B}}}$ , given by  $\mathbb{B} \in \text{ClassA}$ . In subsection 3.1, we prove the spectral gap and in subsection 3.2, we investigate the ground state structure of the Hamiltonians in this class.

### 3.1 The spectral gap of $H_{\Phi_{m,\mathbb{B}}}$ for $\mathbb{B} \in \text{ClassA}$

In this subsection, we prove the following proposition which includes (i),(ii) of Theorem 1.18:

**Proposition 3.1.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ . Let  $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$  and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Then, there exist a constant  $0 < s_{\mathbb{B}} < 1$ , a state  $\varphi_{\mathbb{B}}$ , and a positive element  $e_{\mathbb{B}} \in M_{n_0} \otimes M_{k_R+k_L+1}$ , such that*

(1)  $T_{\mathbb{B}}$  satisfies the Spectral Property II with respect to  $(s_{\mathbb{B}}, e_{\mathbb{B}}, \varphi_{\mathbb{B}})$

(2)  $s(e_{\mathbb{B}}) = \hat{P}_R^{(n_0, k_R, k_L)}$ ,  $s(\varphi_{\mathbb{B}}) = \hat{P}_L^{(n_0, k_R, k_L)}$ .

For all  $N \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ , the map

$$\Gamma_{N,\mathbb{B}}^{(R)}|_{\hat{P}_R^{(n_0, k_R, k_L)}(M_{n_0} \otimes M_{k_R+k_L+1})\hat{P}_L^{(n_0, k_R, k_L)}} : \hat{P}_R^{(n_0, k_R, k_L)}(M_{n_0} \otimes M_{k_R+k_L+1})\hat{P}_L^{(n_0, k_R, k_L)} \rightarrow \mathcal{G}_{N,\mathbb{B}}$$

is bijective and  $m_{\mathbb{B}} \leq 2l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . In particular,  $\ker (H_{\Phi_{m,\mathbb{B}}})_{[0, N-1]} = \text{Ran } \Gamma_{N,\mathbb{B}}^{(R)} = \mathcal{G}_{N,\mathbb{B}}$  is  $n_0^2(k_L + 1)(k_R + 1)$  dimensional for  $m \geq 2l_{\mathbb{B}}$ , and  $N \geq m$ . Furthermore,  $H_{\Phi_{m,\mathbb{B}}}$  is gapped with respect to the open boundary conditions for all  $m \geq \max\{2l_{\mathbb{B}}, \frac{\log(n_0^2(k_L+1)(k_R+1)+1)}{\log n}\}$ .

*Remark 3.2.* We use the notation  $\rho_{\mathbb{B}}$ ,  $x_{\mathbb{B}}$ ,  $y_{\mathbb{B}}$  etc. from Remark 2.10.

**Lemma 3.3.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Define  $\omega_{\mathbb{B}} = (\omega_{1,\mathbb{B}}, \dots, \omega_{n,\mathbb{B}}) \in M_{n_0}^{\times n}$  by*

$$\omega_{\mu,\mathbb{B}} \otimes E_{00}^{(k_R, k_L)} = \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right) B_{\mu} \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right), \quad \mu = 1, \dots, n.$$

Then  $\omega_{\mathbb{B}} \in \text{Prim}_1(n, n_0)$ .

**Proof.** As elements in  $\mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G})$  and  $Y$  are all upper triangular matrices, for any  $l \in \mathbb{N}$  and  $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$ , we have

$$\widehat{\omega_{\mu^{(l)}, \mathbb{B}}} \otimes E_{00}^{(k_R, k_L)} = \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right) \widehat{B_{\mu^{(l)}}} \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right), \quad \mu = 1, \dots, n.$$

Therefore, we have

$$\mathcal{K}_l(\omega_{\mathbb{B}}) \otimes E_{00}^{(k_R, k_L)} = \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right) \mathcal{K}_l(\mathbb{B}) \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right) = M_{n_0} \otimes E_{00}^{(k_R, k_L)},$$

for any  $l \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . This means  $\omega_{\mathbb{B}}$  is primitive. We have to show that  $r_{T_{\omega_{\mathbb{B}}}} = 1$ . First, we have

$$r_{T_{\omega_{\mathbb{B}}}} = \lim_{N \rightarrow \infty} \|T_{\omega_{\mathbb{B}}}^N\|^{\frac{1}{N}} = \lim_{N \rightarrow \infty} \left\| \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right) \left( T_{\mathbb{B}}^N \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right) \right) \left( \mathbb{I} \otimes E_{00}^{(k_R, k_L)} \right) \right\|^{\frac{1}{N}} \leq r_{T_{\mathbb{B}}} = 1.$$

Define  $\mathbb{B}'$  by  $B'_{\mu} := \omega_{\mu, \mathbb{B}} \otimes \Lambda_{\lambda}$ ,  $\mu = 1, \dots, n$ . As  $B_{\mu} - B'_{\mu}$  is in  $M_{n_0} \otimes \text{UT}_{0, k_1+k_2+1}$  and  $B'_{\mu} \in M_{n_0} \otimes \Lambda_{\lambda}$ ,  $T_{\mathbb{B}} - T_{\mathbb{B}'}$  is nilpotent. This and  $B'_{\mu} \in M_{n_0} \otimes \Lambda_{\lambda}$  implies  $\sigma(T_{\mathbb{B}'})^c \subset \sigma(T_{\mathbb{B}})^c$ , i.e., we have  $\sigma(T_{\mathbb{B}}) \subset \sigma(T_{\mathbb{B}'})$ . Therefore, we have  $1 = r_{T_{\mathbb{B}}} \leq r_{T_{\mathbb{B}'}} = r_{T_{\omega_{\mathbb{B}}}}$ , proving  $r_{T_{\omega_{\mathbb{B}}}} = 1$ .  $\square$

**Lemma 3.4.** For  $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$ , we have

$$I_R(D_a) \overline{P_L^{(k_R, k_L)}} = Q_{R, -(a+1)}^{(k_R, k_L)} I_R(D_a) \overline{P_L^{(k_R, k_L)}}, \quad \overline{P_R^{(k_R, k_L)}} I_L(G_b) = \overline{P_R^{(k_R, k_L)}} I_L(G_b) Q_{L, b+1}^{(k_R, k_L)}.$$

**Proof.** Suppose  $E_{ii}^{(k_R, k_L)} I_R(D_a) \overline{P_L^{(k_R, k_L)}} \neq 0$  for  $i \leq 0$ . As the left hand side is equal to  $I_R(E_{ii}^{(k_R, 0)} D_a \sum_{j=-k_R}^{-1} E_{jj}^{(k_R, 0)})$ , this means there is  $j \in \{-k_R, \dots, -1\}$  such that  $\lambda_{R,i} = \lambda_{R,-a} \lambda_{R,j}$ . Note that we have  $|\lambda_{R,j}| < 1$  because  $j \leq -1$ . Therefore, we get  $|\lambda_{R,i}| = |\lambda_{R,-a}| |\lambda_{R,j}| < |\lambda_{R,-a}|$ . This means  $i \leq -a-1$ , proving the first equality. The second one can be proven similarly.  $\square$

We also use the following facts repeatedly.

**Lemma 3.5.** For any  $l \in \mathbb{N}$ ,

$$P_R^{(k_R, k_L)} (1+Y)^l P_L^{(k_R, k_L)} = E_{00}^{(k_R, k_L)}, \quad I_L(G_b) = I_L(G_b) \overline{P_R^{(k_R, k_L)}}, \quad I_R(D_a) = \overline{P_L^{(k_R, k_L)}} I_R(D_a).$$

**Proof.** The first equality follows from (11). The second and the third equations follows from  $G_b \in \text{UT}_{0, k_L+1}$  and  $D_a \in \text{UT}_{0, k_R+1}$ .  $\square$

**Lemma 3.6.** For any  $l \in \mathbb{N}$  and  $\mu^{(l)} \in \{1, \dots, n\}^{\times l}$ , we have

$$\widehat{B_{\mu^{(l)}}} \hat{P}_R^{(n_0, k_R, k_L)} = \hat{P}_R^{(n_0, k_R, k_L)} \widehat{B_{\mu^{(l)}}} \hat{P}_R^{(n_0, k_R, k_L)}, \quad \hat{P}_L^{(n_0, k_R, k_L)} \widehat{B_{\mu^{(l)}}} = \hat{P}_L^{(n_0, k_R, k_L)} \widehat{B_{\mu^{(l)}}} \hat{P}_L^{(n_0, k_R, k_L)}. \quad (41)$$

**Proof.** This is because  $B_{\mu} \in M_{n_0} \otimes \text{UT}_{k_L+k_R+1}$ .  $\square$

**Lemma 3.7.** Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Then the pentad  $(n, n_0(k_L + k_R + 1), \hat{P}_R^{(n_0, k_R, k_L)}, \hat{P}_L^{(n_0, k_R, k_L)}, \mathbb{B})$  satisfies the Condition 2.

**Proof.** There exists  $\omega_{\mathbb{B}} \in \text{Prim}_1(n, n_0)$  given by Lemma 3.3. We would like to show (i)-(viii) of *Condition 2* for the pentad  $(n, n_0(k_R + k_L + 1), \hat{P}_R^{(n_0, k_R, k_L)}, \hat{P}_L^{(n_0, k_R, k_L)}, \mathbb{B})$ . (i): From the definition of  $\hat{P}_R^{(n_0, k_R, k_L)}, \hat{P}_L^{(n_0, k_R, k_L)}$ , we have  $\hat{P}_R^{(n_0, k_R, k_L)} \hat{P}_L^{(n_0, k_R, k_L)} = \hat{P}_L^{(n_0, k_R, k_L)} \hat{P}_R^{(n_0, k_R, k_L)} = \hat{E}_{00}^{(k_R, k_L)} \neq 0$ . (ii) and (iii) are Lemma 3.6. (iv): From  $\omega_{\mathbb{B}} \in \text{Prim}_1(n, n_0)$  and Lemma C.6, (iv) can be checked. (v),(vi): Note, as in the proof of Lemma 3.3, that  $r_{T_{\mathbb{B}'}} \leq r_{T_{\mathbb{B}'}} \leq |\lambda_{-1}|^2$ , where  $\mathbb{B}' = (B'_{\mu})_{\mu}$

is given by  $B'_{\mu} = \omega_{\mu, \mathbb{B}} \otimes \Lambda_{\lambda} \overline{P_L^{(k_R, k_L)}}$ . This implies (v). (vi) can be shown similarly.

(vii),(viii) : We prove (viii). The proof for (vii) is the same. Assume that  $k_R \geq 1$ . Let  $l \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$  and  $\eta \in \hat{P}_R^{(n_0, k_R, k_L)}(\mathbb{C}^{n_0} \otimes \mathbb{C}^{k_R + k_L + 1})$  with  $\hat{P}_R^{(n_0, k_R, k_L)} \hat{P}_L^{(n_0, k_R, k_L)} \eta = \hat{E}_{00}^{(k_R, k_L)} \eta \neq 0$ . Fix some  $\alpha \in \{1, \dots, n_0\}$  such that  $(e_{\alpha, \alpha}^{(n_0)} \otimes E_{00}^{(k_R, k_L)}) \eta \neq 0$ . Set  $\eta' := (\mathbb{I} \otimes \Lambda_{\lambda}^l (1 + Y)^l) \eta$ . Note that

$$\mathcal{K}_l(\mathbb{B}_{\hat{P}_R^{(n_0, k_R, k_L)}}) \eta = \mathcal{K}_l(\mathbb{B}) \hat{P}_R^{(n_0, k_R, k_L)} \eta = \left( M_{n_0} \otimes \text{span} \left\{ P_R^{(k_R, k_L)}, I_R(D_a) \right\} \right) \eta',$$

and  $\langle \chi_{\alpha}^{(n_0)} \otimes f_0^{(k_R, k_L)}, \eta' \rangle \neq 0$ . Here we used Lemma 3.5 and Lemma 3.6, for the first equality, and (17) for the second one.

We consider the following proposition for  $i = -k_R, \dots, 0$ :

$$(P_i): \mathbb{C}^{n_0} \otimes \left( Q_{R,i}^{(k_R, k_L)} \mathbb{C}^{k_R + k_L + 1} \right) \subset \mathcal{K}_l(\mathbb{B}_{\hat{P}_R^{(n_0, k_R, k_L)}}) \eta.$$

To see that  $(P_{-k_R})$  holds, note that for any  $\beta \in \{1, \dots, n_0\}$  we have  $(e_{\beta\alpha}^{(n_0)} \otimes I_R(D_{k_R})) \eta' \in \mathcal{K}_l(\mathbb{B}_{\hat{P}_R^{(n_0, k_R, k_L)}}) \eta$ . By Lemma 3.4, we have  $I_R(D_{k_R}) = I_R(D_{k_R}) E_{00}^{(k_R, k_L)} = E_{-k_R, 0}^{(k_R, k_L)}$ . Therefore, we have

$$(e_{\beta\alpha}^{(n_0)} \otimes I_R(D_{k_R})) \eta' = \langle \chi_{\alpha}^{(n_0)} \otimes f_0^{(k_R, k_L)}, \eta' \rangle \chi_{\beta}^{(n_0)} \otimes f_{-k_R}^{(k_R, k_L)},$$

with  $\langle \chi_{\alpha}^{(n_0)} \otimes f_0^{(k_R, k_L)}, \eta' \rangle \neq 0$ . Hence we have  $\chi_{\beta}^{(n_0)} \otimes f_{-k_R}^{(k_R, k_L)} \in \mathcal{K}_l(\mathbb{B}_{\hat{P}_R^{(n_0, k_R, k_L)}}) \eta$ , proving  $(P_{-k_R})$ .

Suppose that  $(P_{i-1})$  holds for some  $i \leq 0$ . We show that  $(P_i)$  holds. To see this, note that for any  $\beta \in \{1, \dots, n_0\}$  we have  $(e_{\beta\alpha}^{(n_0)} \otimes I_R(D_{-i})) \eta' \in \mathcal{K}_l(\mathbb{B}_{\hat{P}_R^{(n_0, k_R, k_L)}}) \eta$ , if  $i \leq -1$ , and  $(e_{\beta\alpha}^{(n_0)} \otimes I_R(\mathbb{I})) \eta' \in \mathcal{K}_l(\mathbb{B}_{\hat{P}_R^{(n_0, k_R, k_L)}}) \eta$ . Now if  $i \leq -1$ , by Lemma 3.4, we have

$$I_R(D_{-i}) = I_R(D_{-i}) E_{00}^{(k_R, k_L)} + I_R(D_{-i}) Q_{R,-1}^{(k_R, k_L)} = E_{i0}^{(k_R, k_L)} + Q_{R,i-1}^{(k_R, k_L)} I_R(D_{-i}) Q_{R,-1}^{(k_R, k_L)}.$$

This relation also holds for  $i = 0$  case, if we set  $D_0 = \mathbb{I}$ . Therefore, we have

$$(e_{\beta\alpha}^{(n_0)} \otimes I_R(D_{-i})) \eta' = \langle \chi_{\alpha}^{(n_0)} \otimes f_0^{(k_R, k_L)}, \eta' \rangle \chi_{\beta}^{(n_0)} \otimes f_i^{(k_R, k_L)} + \text{an element of } \mathbb{C}^{n_0} \otimes Q_{R,i-1}^{(k_R, k_L)} \mathbb{C}^{k_R + k_L + 1}$$

with  $\langle \chi_{\alpha}^{(n_0)} \otimes f_0^{(k_R, k_L)}, \eta' \rangle \neq 0$ . From  $(P_{i-1})$ , we have  $\mathbb{C}^{n_0} \otimes Q_{R,i-1}^{(k_R, k_L)} \mathbb{C}^{k_R + k_L + 1} \subset \mathcal{K}_l(\mathbb{B}_{\hat{P}_R^{(n_0, k_R, k_L)}}) \eta$ .

Hence we have  $\chi_{\beta}^{(n_0)} \otimes f_i^{(k_R, k_L)} \in \mathcal{K}_l(\mathbb{B}_{\hat{P}_R^{(n_0, k_R, k_L)}}) \eta$ , proving  $(P_i)$ .

Inductively, we obtain  $(P_0)$ , proving (viii) for  $k_R \geq 1$  case. The  $k_R = 0$  case is much simpler. We just need to note  $e_{\beta\alpha}^{(n_0)} \eta \in \mathcal{K}_l(\mathbb{B}) \eta$ .  $\square$

**Lemma 3.8.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$  and  $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$ . Let  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Then the pentad  $(n, n_0(k_R + k_L + 1), \hat{P}_R^{(n_0, k_R, k_L)}, \hat{P}_L^{(n_0, k_R, k_L)}, \mathbb{B})$  satisfies the *Condition 3* for  $l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ .

**Proof.** Note that the matrices  $\mathbb{I}$ ,  $\{I_R^{(k_R, k_L)}(D_a)\}_{a=1}^{k_R}$ ,  $\{I_L^{(k_R, k_L)}(G_b)\}_{b=1}^{k_L}$   $\left\{E_{-a,b}^{(k_R, k_L)}\right\}_{a=1, \dots, k_R, b=1, \dots, k_L}$  are linearly independent. By this fact, for  $l \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ , we have

$$\begin{aligned} \dim \mathcal{K}_l(\mathbb{B}) &= \dim \left( \left( M_{n_0} \otimes \left( \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) (\Lambda_{\lambda} (1+Y))^l \right) \right) \right) \\ &= n_0^2 (k_R + 1) (k_L + 1) = \left( \text{rank } \hat{P}_R^{(n_0, k_R, k_L)} \right) \cdot \left( \text{rank } \hat{P}_L^{(n_0, k_R, k_L)} \right). \end{aligned}$$

□

**Lemma 3.9.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$  and  $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$ . Let  $\mathbb{B} \in \text{ClassA}$  with respect to  $(\lambda, \mathbb{D}, \mathbb{G}, Y)$ . Then the triple  $(n, n_0(k_R + k_L + 1), \mathbb{B})$  satisfies the Condition 4 for  $(l_{\mathbb{B}}, l_{\mathbb{B}})$ .

**Proof.** Set  $l_{\mathbb{B}} := l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . By the definition of  $l_{\mathbb{B}}$  the invertible element  $\mathbb{I} \otimes \Lambda_{\lambda}^{l_{\mathbb{B}}}(1+Y)^{l_{\mathbb{B}}}$  belongs to  $\left( M_{n_0} \otimes \left( \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) (\Lambda_{\lambda} (1+Y))^{l_{\mathbb{B}}} \right) \right) = \mathcal{K}_{l_{\mathbb{B}}}(\mathbb{B})$ . We also note by (16),

$$\begin{aligned} \left( \mathbb{I} \otimes (\Lambda_{\lambda}(1+Y))^{l_{\mathbb{B}}} \right)^{-1} \mathcal{K}_{N+l_{\mathbb{B}}}(\mathbb{B}) &= \left( \mathbb{I} \otimes (\Lambda_{\lambda}(1+Y))^{l_{\mathbb{B}}} \right)^{-1} \left( M_{n_0} \otimes \left( \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) (\Lambda_{\lambda} (1+Y))^{N+l_{\mathbb{B}}} \right) \right) \\ &\subset \left( M_{n_0} \otimes \left( \mathcal{D}(k_R, k_L, \mathbb{D}, \mathbb{G}) (\Lambda_{\lambda} (1+Y))^N \right) \right) = \mathcal{K}_N(\mathbb{B}), \quad N \geq l_{\mathbb{B}}. \end{aligned} \quad (42)$$

□

**Lemma 3.10.** Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$  and  $(\lambda, \mathbb{D}, \mathbb{G}, Y) \in \mathcal{T}(k_R, k_L)$ . Let  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Then we have  $M_{\mathbb{B}, \hat{P}_R^{(n_0, k_R, k_L)}, \hat{P}_L^{(n_0, k_R, k_L)}} \leq l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ .

**Proof.** Let  $l \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Then, we have

$$\hat{P}_R^{(n_0, k_R, k_L)} \mathcal{K}_l(\mathbb{B}) \hat{P}_L^{(n_0, k_R, k_L)} = M_{n_0} \otimes \left( P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)} \right). \quad (43)$$

To see this, note that from Lemma 3.5, we have

$$\begin{aligned} P_R^{(k_R, k_L)} (\Lambda_{\lambda} (1+Y))^l P_L^{(k_R, k_L)} &= E_{00}^{(k_R, k_L)}, \\ P_R^{(k_R, k_L)} I_R^{(k_R, k_L)}(D_a) (\Lambda_{\lambda} (1+Y))^l P_L^{(k_R, k_L)} &= I_R^{(k_R, k_L)}(D_a E_{00}^{(k_R, 0)}) = I_R^{(k_R, k_L)}(E_{-a0}^{(k_R, 0)}) = E_{-a,0}^{(k_R, k_L)}. \end{aligned}$$

Furthermore, for any  $i \in \{-k_R, \dots, 0\}$ , we have

$$E_{ib}^{(k_R, k_L)} (\Lambda_{\lambda} (1+Y))^l = \sum_{j'=1}^{k_L} \left\langle f_b^{(k_R, k_L)}, (\Lambda_{\lambda} (1+Y))^l f_{j'}^{(k_R, k_L)} \right\rangle E_{ij'}^{(k_R, k_L)}, \quad b \in \{1, \dots, k_L\},$$

$$E_{ij}^{(k_R, k_L)} = \sum_{b'=1}^{k_L} \left\langle f_j^{(k_R, k_L)}, (\Lambda_{\lambda} (1+Y))^{-l} f_{b'}^{(k_R, k_L)} \right\rangle E_{ib'}^{(k_R, k_L)} (\Lambda_{\lambda} (1+Y))^l, \quad j \in \{1, \dots, k_L\}.$$

In particular, we have  $\text{span}\{E_{ib}^{(k_R, k_L)} (\Lambda_{\lambda} (1+Y))^l\}_{b=1}^{k_L} = \text{span}\{E_{ij}^{(k_R, k_L)}\}_{j=1}^{k_L}$ , and we obtain

$$\begin{aligned} \text{span}\left\{P_R^{(k_R, k_L)} I_L^{(k_R, k_L)}(G_b) (\Lambda_{\lambda} (1+Y))^l\right\}_{b=1}^{k_L} &= \text{span}\left\{E_{0b}^{(k_R, k_L)} (\Lambda_{\lambda} (1+Y))^l\right\}_{b=1}^{k_L} = \text{span}\left\{E_{0j}^{(k_R, k_L)}\right\}_{j=1}^{k_L}, \\ \text{span}\left\{E_{-a,b}^{(k_R, k_L)} (\Lambda_{\lambda} (1+Y))^l\right\}_{b=1}^{k_L} &= \text{span}\left\{E_{-a,j}^{(k_R, k_L)}\right\}_{j=1}^{k_L}, \quad a = 1, \dots, k_R. \end{aligned}$$

These proves (43). If  $X \in \left( \hat{P}_R^{(n_0, k_R, k_L)} (M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_L^{(n_0, k_R, k_L)} \right) \cap \ker \Gamma_{l, \mathbb{B}}^{(R)}$ , then (43) implies  $X = 0$ , proving the claim. □

**Proof of Proposition 3.1.** The first statement follows from Lemma 3.7 and Lemma 2.9. The bijectivity of  $\Gamma_{N,\mathbb{B}}^{(R)}$  for  $N \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$  can be seen from Lemma 3.10. From Lemma 3.7, 3.8 and Lemma 3.9, applying Proposition 2.6 we get  $m_{\mathbb{B}} \leq 2l_{\mathbb{B}}$  and  $\{\mathcal{G}_{l,\mathbb{B}}\}$  satisfies *Condition 1*. From Theorem 2.2, for all  $m \geq 2l_{\mathbb{B}}$  and  $N \geq m$ , we have  $\ker (H_{\Phi_{m,\mathbb{B}}})_{[0,N-1]} = \mathcal{G}_{N,\mathbb{B}} = \text{Ran } \Gamma_{N,\mathbb{B}}^{(R)}$  and from the bijectivity proven above, its dimension is  $n_0^2(k_L + 1)(k_R + 1)$ . Furthermore, if  $m \geq \max\{2l_{\mathbb{B}}, \frac{\log(n_0^2(k_L+1)(k_R+1)+1)}{\log n}\}$ , then  $G_m \neq \mathbb{I}$  and  $\gamma_{l,m} > 0$  for any  $l \geq \max\{l_0, m\}$ . Therefore, from (2) of Theorem 2.2,  $H_{\Phi_{m,\mathbb{B}}}$  is gapped with respect to the open boundary conditions.  $\square$

### 3.2 Edge states of $H_{\Phi_{m,\mathbb{B}}}$

In this subsection, we prove (iii),(iv),(v),(vi),(vii), (viii) of Theorem 1.18.

**Lemma 3.11.** *Let  $n, n_0 \in \mathbb{N}$  with  $n \geq 2$ ,  $k_R, k_L \in \mathbb{N} \cup \{0\}$ ,  $\mathbb{B} \in \text{ClassA}$ , and  $m_{\mathbb{B}} \leq m \in \mathbb{N}$ . Let  $\Gamma = \mathbb{Z}$ ,  $\Gamma = (-\infty, -1]$  or  $\Gamma = [0, \infty)$ . For a state  $\omega$  on  $\mathcal{A}_{\Gamma}$ ,  $\omega \in \mathcal{S}_{\Gamma}(H_{\Phi_{m,\mathbb{B}}})$  if and only if  $\omega(\tau_i(1 - G_{m,\mathbb{B}})) = 0$  for any  $i \in \mathbb{Z}$  with  $[i, i + m - 1] \subset \Gamma$ . In particular,  $\mathcal{S}_{\Gamma}(H_{\Phi_{m,\mathbb{B}}})$  is a convex set.*

**Proof.** Suppose that  $\omega$  is a state on  $\mathcal{A}_{\Gamma}$  such that  $\omega(\tau_i(1 - G_{m,\mathbb{B}})) = 0$ , for any  $i \in \mathbb{Z}$  with  $[i, i + m - 1] \subset \Gamma$ . Then its restriction  $\omega|_{\mathcal{A}_I}$  to each interval  $I \subset \Gamma$  is a ground state of  $(H_{\Phi_{m,\mathbb{B}}})_I$ . Hence  $\omega$  is a  $wk^*$ -accumulation point of extensions of  $\omega|_{\mathcal{A}_I} \in \mathcal{S}_I(H_{\Phi_{m,\mathbb{B}}})$ , hence  $\omega \in \mathcal{S}_{\Gamma}(H_{\Phi_{m,\mathbb{B}}})$ , by definition.

If  $\omega \in \mathcal{S}_{\Gamma}(H_{\Phi_{m,\mathbb{B}}})$ , then there exists a subnet  $\{I'\}$  of intervals in  $\Gamma$  associated with states  $\omega_{I'}$  on  $\mathcal{A}_{\Gamma}$  such that  $\omega_{I'}|_{\mathcal{A}_{I'}} \in \mathcal{S}_{I'}(H_{\Phi_{m,\mathbb{B}}})$ , and  $\omega = \text{wk}^* - \lim_{I'} \omega_{I'}$ . Because  $m \geq m_{\mathbb{B}}$ , we have  $\omega \circ \tau_i(1 - G_{m,\mathbb{B}}) = \lim_{I'} \omega_{I'}(\tau_i(1 - G_{m,\mathbb{B}})) = 0$ , for any  $i \in \mathbb{Z}$  with  $[i, i + m - 1] \subset \Gamma$ .  $\square$

**Lemma 3.12.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Set*

$$\mathbb{L}_{\mathbb{B}}(A) := \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\} \times l} \left\langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \right\rangle \left( \widehat{B_{\nu^{(l)}}} \right)^* \rho_{\mathbb{B}} \widehat{B_{\mu^{(l)}}},$$

if  $A \in \mathcal{A}_{[-l, -1]} \simeq \otimes_{i=0}^{l-1} M_n$  for  $l \in \mathbb{N}$ . Then  $\mathbb{L}_{\mathbb{B}}$  defines a well-defined completely positive map on  $\mathcal{A}_{(-\infty, -1]}^{\text{loc}}$ . This  $\mathbb{L}_{\mathbb{B}}$  extends to a completely positive map from the half-infinite chain  $\mathcal{A}_{(-\infty, -1]}$  to  $M_{n_0} \otimes M_{k_R+k_L+1}$ , which we will denote by the same symbol  $\mathbb{L}_{\mathbb{B}}$ . We have

$$\text{Ran } \mathbb{L}_{\mathbb{B}} = \hat{P}_L^{(n_0, k_R, k_L)} (M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_L^{(n_0, k_R, k_L)}. \quad (44)$$

**Proof.** Note that from the relation  $\sum_{\mu=1}^n B_{\mu}^* \rho_{\mathbb{B}} B_{\mu} = \rho_{\mathbb{B}}$ ,  $\mathbb{L}_{\mathbb{B}}$  is well-defined. It can be checked directly that  $\mathbb{L}_{\mathbb{B}}|_{\mathcal{A}_{[-l, -1]}}$  defines a completely positive map on  $\mathcal{A}_{[-l, -1]}$  with norm  $\|\mathbb{L}_{\mathbb{B}}|_{\mathcal{A}_{[-l, -1]}}\| = \|\mathbb{L}_{\mathbb{B}}(1)\| = \|\rho_{\mathbb{B}}\|$ , for any  $l \in \mathbb{N}$ . Therefore, we can extend it to a completely positive map on  $\mathcal{A}_{(-\infty, -1]}$ .

Next we check (44). The inclusion  $\text{Ran } \mathbb{L}_{\mathbb{B}} \subset \hat{P}_L^{(n_0, k_R, k_L)} (M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_L^{(n_0, k_R, k_L)}$  can be seen from  $\left( \widehat{B_{\nu^{(l)}}} \right)^* \rho_{\mathbb{B}} \widehat{B_{\mu^{(l)}}} \in \hat{P}_L^{(n_0, k_R, k_L)} (M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_L^{(n_0, k_R, k_L)}$  which follows from (41) and  $s(\rho_{\mathbb{B}}) = \hat{P}_L^{(n_0, k_R, k_L)}$ .

To see the opposite inclusion, note that for any  $l \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ ,  $\alpha, \beta \in \{1, \dots, n_0\}$ , and  $b, b' = 0, \dots, k_L$ , we have

$$\begin{aligned} & \left( e_{1,\alpha}^{(n_0)} \otimes (\Lambda_{\lambda} (1 + Y))^l I_L(G_b) \right)^* \rho_{\mathbb{B}} \left( e_{1,\beta}^{(n_0)} \otimes (\Lambda_{\lambda} (1 + Y))^l I_L(G_{b'}) \right) \\ &= \left\langle \chi_1^{(n_0)} \otimes f_0^{(k_R, k_L)}, \rho_{\mathbb{B}} \left( \chi_1^{(n_0)} \otimes f_0^{(k_R, k_L)} \right) \right\rangle e_{\alpha, \beta}^{(n_0)} \otimes E_{b, b'}^{(k_R, k_L)} \\ &+ \text{an element of } M_{n_0} \otimes \left( Q_{L, b+1}^{(k_R, k_L)} M_{k_R+k_L+1} Q_{L, b'}^{(k_R, k_L)} + Q_{L, b}^{(k_R, k_L)} M_{k_R+k_L+1} Q_{L, b'+1}^{(k_R, k_L)} \right), \end{aligned}$$



by Lemma 3.4. Here,  $G_0$  should be understood as  $G_0 := \mathbb{I}$ . As  $l \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y)$ , the left hand side is in  $\mathcal{K}_l(\mathbb{B})^* \rho_{\mathbb{B}} \mathcal{K}_l(\mathbb{B})$ . Because  $s(\rho_{\mathbb{B}}) = \hat{P}_L^{(n_0, k_R, k_L)} \left\langle \chi_1^{(n_0)} \otimes f_0^{(k_R, k_L)}, \rho_{\mathbb{B}} \left( \chi_1^{(n_0)} \otimes f_0^{(k_R, k_L)} \right) \right\rangle$  is not zero. Therefore, we have

$$e_{\alpha, \beta}^{(n_0)} \otimes E_{b, b'}^{(k_R, k_L)} \in \mathcal{K}_l(\mathbb{B})^* \rho_{\mathbb{B}} \mathcal{K}_l(\mathbb{B}) + M_{n_0} \otimes \left( Q_{L, b+1}^{(k_R, k_L)} M_{k_R+k_L+1} Q_{L, b'}^{(k_R, k_L)} + Q_{L, b}^{(k_R, k_L)} M_{k_R+k_L+1} Q_{L, b'+1}^{(k_R, k_L)} \right)$$

for any  $l \geq l_{\mathbb{B}}(n, n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y)$ ,  $\alpha, \beta \in \{1, \dots, n_0\}$ , and  $b, b' = 0, \dots, k_L$ . In particular, we have  $e_{\alpha, \beta}^{(n_0)} \otimes E_{k_L, k_L}^{(k_R, k_L)} \in \mathcal{K}_l(\mathbb{B})^* \rho_{\mathbb{B}} \mathcal{K}_l(\mathbb{B})$ . Starting from this, by induction with respect to  $b, b'$ , we conclude

$$M_{n_0} \otimes \hat{P}_L^{(n_0, k_R, k_L)} (M_{k_R+k_L+1}) \hat{P}_L^{(n_0, k_R, k_L)} \subset \text{span}(\mathcal{K}_l(\mathbb{B})^* \rho_{\mathbb{B}} \mathcal{K}_l(\mathbb{B})) \subset \text{Ran } \mathbb{L}_{\mathbb{B}}.$$

□

Similarly, we obtain the following.

**Lemma 3.13.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y)$ . Set*

$$\mathbb{R}_{\mathbb{B}}(A) := \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\} \times l} \left\langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \right\rangle \widehat{B_{\mu^{(l)}}} e_{\mathbb{B}} \left( \widehat{B_{\nu^{(l)}}} \right)^*,$$

if  $A \in \mathcal{A}_{[0, l-1]} \simeq \otimes_{i=0}^{l-1} M_n$  for  $l \in \mathbb{N}$ . Then  $\mathbb{R}_{\mathbb{B}}$  defines a well-defined completely positive map on  $\mathcal{A}_{[0, \infty)}^{\text{loc}}$ . This  $\mathbb{R}_{\mathbb{B}}$  extends to a completely positive map from the half-infinite chain  $\mathcal{A}_{[0, \infty)}$  to  $M_{n_0} \otimes M_{k_R+k_L+1}$ , which we will denote by the same symbol  $\mathbb{R}_{\mathbb{B}}$ . We have

$$\text{Ran } \mathbb{R}_{\mathbb{B}} = \hat{P}_R^{(n_0, k_R, k_L)} (M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_R^{(n_0, k_R, k_L)}.$$

**Lemma 3.14.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y)$ . For each  $\sigma_L \in \mathfrak{E}_{n_0(k_L+1)}$ , under the identification  $M_{n_0(k_L+1)} \simeq M_{n_0} \otimes P_L^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ , define  $\Xi_L(\sigma_L) : \mathcal{A}_{(-\infty, -1]} \rightarrow \mathbb{C}$  by*

$$\Xi_L(\sigma_L)(A) := \sigma_L(y_{\mathbb{B}}^{\frac{1}{2}} \mathbb{L}_{\mathbb{B}}(A) y_{\mathbb{B}}^{\frac{1}{2}}), \quad A \in \mathcal{A}_{(-\infty, -1]}.$$

For each  $\sigma_R \in \mathfrak{E}_{n_0(k_R+1)}$ , under the identification  $M_{n_0(k_R+1)} \simeq M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_R^{(k_R, k_L)}$ , define  $\Xi_R(\sigma_R) : \mathcal{A}_{[0, \infty)} \rightarrow \mathbb{C}$  by

$$\Xi_R(\sigma_R)(A) := \sigma_R(x_{\mathbb{B}}^{\frac{1}{2}} \mathbb{R}_{\mathbb{B}}(A) x_{\mathbb{B}}^{\frac{1}{2}}), \quad A \in \mathcal{A}_{[0, \infty)}.$$

For each nonzero  $X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ , set

$$\sigma_{L, X} := \frac{\text{Tr} \left( \rho_{\mathbb{B}}^{\frac{1}{2}} X^* e_{\mathbb{B}} X \rho_{\mathbb{B}}^{\frac{1}{2}} \right)}{\varphi_{\mathbb{B}}(X^* e_{\mathbb{B}} X)} \in \mathcal{E}_{n_0(k_L+1)}, \quad \sigma_{R, X} := \frac{\varphi_{\mathbb{B}} \left( X^* e_{\mathbb{B}}^{\frac{1}{2}} (\cdot) e_{\mathbb{B}}^{\frac{1}{2}} X \right)}{\varphi_{\mathbb{B}}(X^* e_{\mathbb{B}} X)} \in \mathcal{E}_{n_0(k_R+1)}. \quad (45)$$

Then we have

$$\begin{aligned} & \left| \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(X), \tau_{N-l}(A) \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\rangle - \varphi_{\mathbb{B}}(X^* e_{\mathbb{B}} X) \Xi_L(\sigma_{L, X}) \circ \tau_{-l}(A) \right| \leq \tilde{E}_{\mathbb{B}}(N-l) F_{\mathbb{B}} \|X\|^2 \|A\|, \\ & \left| \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(X), A \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\rangle - \varphi_{\mathbb{B}}(X^* e_{\mathbb{B}} X) \Xi_R(\sigma_{R, X})(A) \right| \leq \tilde{E}_{\mathbb{B}}(N-l) F_{\mathbb{B}} \|X\|^2 \|A\| \end{aligned} \quad (46)$$

for all nonzero  $X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ ,  $l \in \mathbb{N}$ ,  $A \in \mathcal{A}_{[0, l-1]}$ , and  $l \leq N$ .

*Remark 3.15.* Recall the definitions (21).

**Proof.** Let  $A \in \mathcal{A}_{[0, l-1]}$  for some  $l \in \mathbb{N}$  and  $l \leq N$ . Define a linear map  $V : \left( \bigotimes_{i=0}^{l-1} \mathbb{C}^n \right) \otimes (\mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+k_R+1}) \rightarrow \mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+k_R+1}$  by

$$V(\xi \otimes \eta) := \sum_{\mu^{(l)} \in \{1, \dots, n\}^l} \left\langle \widehat{\psi_{\mu^{(l)}}}, \xi \right\rangle B_{\mu^{(l)}} \eta, \quad \xi \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n, \quad \eta \in \mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+k_R+1}.$$

We define a linear map  $\Theta_A : M_{n_0} \otimes M_{k_R+k_L+1} \rightarrow M_{n_0} \otimes M_{k_R+k_L+1}$  by

$$\Theta_A(X) = V(A \otimes X)V^*, \quad X \in M_{n_0} \otimes M_{k_R+k_L+1}.$$

Note that

$$\varphi_{\mathbb{B}} \circ \Theta_A = \text{Tr}(L_{\mathbb{B}} \circ \tau_{-l}(A)(\cdot)), \quad \Theta_A(e_{\mathbb{B}}) = R_{\mathbb{B}}(A).$$

Then we have

$$\|\Theta_A\| \leq \|A\| \|V\| \|V^*\| = \|A\| \|VV^*\| = \|A\| \|T_{\mathbb{B}}^l(1)\| \leq \|A\| F_{\mathbb{B}}.$$

(Recall the estimate (22).) Using these notations, for nonzero  $X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ , we have

$$\begin{aligned} & \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(X), \tau_{N-l}(A) \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\rangle \\ &= \sum_{\mu^{(l)}, \nu^{(l)}} \sum_{\alpha, \beta=1}^{n_0} \sum_{i, j=-k_R}^{k_L} \left\langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \right\rangle \\ & \left\langle \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)}, T_{\mathbb{B}}^{N-l} \left( \widehat{B_{\mu^{(l)}}} X^* \left( \left| \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)} \right\rangle \left\langle \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)} \right| \right) X \left( \widehat{B_{\nu^{(l)}}} \right)^* \right) \left( \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)} \right) \right\rangle \\ &= \varphi_{\mathbb{B}}(X^* e_{\mathbb{B}} X) \Xi_L(\sigma_{L, X}) \circ \tau_{-l}(A) \\ &+ \sum_{\alpha, \beta=1}^{n_0} \sum_{i, j=-k_R}^{k_L} \\ & \left\langle \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)}, T_{\mathbb{B}}^{N-l} \circ \left( \mathbb{I} - P_{\{1\}}^{T_{\mathbb{B}}} \right) \circ \Theta_A \left( X^* \left( \left| \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)} \right\rangle \left\langle \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)} \right| \right) X \right) \left( \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)} \right) \right\rangle \end{aligned}$$

Hence we have

$$\left| \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(X), \tau_{N-l}(A) \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\rangle - \varphi_{\mathbb{B}}(X^* e_{\mathbb{B}} X) \Xi_L(\sigma_{L, X}) \circ \tau_{-l}(A) \right| \leq \tilde{E}_{\mathbb{B}}(N-l) F_{\mathbb{B}} \|X\|^2 \|A\|$$

The second inequality can be proven similarly.  $\square$

**Lemma 3.16.** Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Let  $\sigma_L \in \mathfrak{E}_{n_0(k_L+1)}$ ,  $\sigma_R \in \mathfrak{E}_{n_0(k_R+1)}$ , and consider  $\Xi_L(\sigma_L)$  and  $\Xi_R(\sigma_R)$  defined in Lemma 3.14. Then for  $m \geq m_{\mathbb{B}}$ , we have  $\Xi_L(\sigma_L) \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}})$ , and  $\Xi_R(\sigma_R) \in \mathcal{S}_{[0, \infty)}(H_{\Phi_{m, \mathbb{B}}})$ . For  $m \geq m_{\mathbb{B}}$ , the maps  $\Xi_L : \mathfrak{E}_{n_0(k_L+1)} \rightarrow \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}})$  and  $\Xi_R : \mathfrak{E}_{n_0(k_R+1)} \rightarrow \mathcal{S}_{[0, \infty)}(H_{\Phi_{m, \mathbb{B}}})$  are affine bijections.

**Proof.** Let  $\sigma_L \in \mathfrak{E}_{n_0(k_L+1)}$ . As  $L_{\mathbb{B}}$  is positive,  $\Xi_L(\sigma_L)$  is positive. Furthermore, we have

$$\Xi_L(\sigma_L)(1) = \sigma_L(y_{\mathbb{B}}^{\frac{1}{2}} \mathbb{I}_{\mathbb{B}}(1) y_{\mathbb{B}}^{\frac{1}{2}}) = \sigma_L(y_{\mathbb{B}}^{\frac{1}{2}} \rho_{\mathbb{B}} y_{\mathbb{B}}^{\frac{1}{2}}) = \sigma_L(1) = 1.$$

Hence  $\Xi_L(\sigma_L)$  is a state on  $A \in \mathcal{A}_{(-\infty, -1]}$ .

Next we show  $\Xi_L(\sigma_L) \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}})$ , for  $m \geq \mathbf{m}_{\mathbb{B}}$ . From Lemma 3.11, it suffices to show  $\Xi_L(\sigma_L)(\tau_i(1 - G_{m, \mathbb{B}})) = 0$  for any  $i \in \mathbb{Z}$  with  $[i, i + m - 1] \subset (-\infty, -1]$ . To do this, note that there exists a set  $\{Z_j\}_{j=1}^{n_0(k_L+1)} \subset M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  such that  $\sum_j \text{Tr}(e_{\mathbb{B}}(Z_j \cdot Z_j^*)) = \sigma_L(y_{\mathbb{B}}^{\frac{1}{2}} \cdot y_{\mathbb{B}}^{\frac{1}{2}})$ . From Lemma 3.14, we have

$$\begin{aligned} \Xi_L(\sigma_L)(\tau_i(1 - G_{m, \mathbb{B}})) &= \sum_j \text{Tr}(e_{\mathbb{B}}(Z_j(\mathbb{L}_{\mathbb{B}}(\tau_i(1 - G_{m, \mathbb{B}})))Z_j^*)) \\ &= \sum_j \lim_{N \rightarrow \infty} \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(Z_j), \tau_N(\tau_i(1 - G_{m, \mathbb{B}})) \Gamma_{N, \mathbb{B}}^{(R)}(Z_j) \right\rangle = 0, \end{aligned}$$

because  $m \geq \mathbf{m}_{\mathbb{B}}$ .

It is clear from the definition that  $\Xi_L$  is affine. To show that  $\Xi_L$  is injective, assume that  $\Xi_L(\sigma_1) = \Xi_L(\sigma_2)$  for some  $\sigma_1, \sigma_2 \in \mathfrak{E}_{n_0(k_L+1)}$ . Then we have

$$\sigma_1(y_{\mathbb{B}}^{\frac{1}{2}} \mathbb{L}_{\mathbb{B}}(A) y_{\mathbb{B}}^{\frac{1}{2}}) = \Xi_L(\sigma_1)(A) = \Xi_L(\sigma_2)(A) = \sigma_2(y_{\mathbb{B}}^{\frac{1}{2}} \mathbb{L}_{\mathbb{B}}(A) y_{\mathbb{B}}^{\frac{1}{2}}),$$

for all  $A \in \mathcal{A}_{(-\infty, -1]}$ . As  $\mathbb{L}_{\mathbb{B}}$  is onto  $\hat{P}_L^{(n_0, k_R, k_L)}(M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_L^{(n_0, k_R, k_L)} \simeq M_{n_0(k_L+1)}$  and  $y_{\mathbb{B}}^{\frac{1}{2}}$  is an invertible element of  $\hat{P}_L^{(n_0, k_R, k_L)}(M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_L^{(n_0, k_R, k_L)} \simeq M_{n_0(k_L+1)}$ , we have  $\sigma_1 = \sigma_2$ .

To complete the proof, we prove that  $\Xi_L$  is onto  $\mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}})$  for  $m \geq \mathbf{m}_{\mathbb{B}}$ . Let  $\omega \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}})$ . For each  $N \in \mathbb{N}$ , let  $D_N$  be the density matrix of the restriction of  $\omega$  to  $\mathcal{A}_{[-N, -1]}$ , namely  $\omega(A) = \text{Tr}_{[-N, -1]}(D_N A)$  for any  $A \in \mathcal{A}_{[-N, -1]}$ . By Lemma 3.11, we have  $\omega(\tau_i(1 - G_{m, \mathbb{B}})) = 0$  for any  $i \in \mathbb{Z}$  with  $[i, i + m - 1] \subset (-\infty, -1]$ . Therefore, from the intersection property, we have that  $\text{Ran}(\tau_N(D_N)) \subset \mathcal{G}_{N, \mathbb{B}}$  for all  $N \geq \mathbf{m}_{\mathbb{B}}$ .

From Proposition 3.1, this means for  $N \geq \max\{l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}), \mathbf{m}_{\mathbb{B}}\}$ , that there exist

$$X_{i, N} \in \hat{P}_R^{(n_0, k_R, k_L)}(M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_L^{(n_0, k_R, k_L)}, \quad i = 1, \dots, n_0^2(k_R + 1)(k_L + 1),$$

such that

$$\tau_N(D_N) = \sum_i \left| \Gamma_{N, \mathbb{B}}^{(R)}(X_{i, N}) \right\rangle \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(X_{i, N}) \right|.$$

Furthermore, for  $N \geq L_{\mathbb{B}}$ , we have

$$\sum_{i=1}^{n_0^2(k_R+1)(k_L+1)} \|X_{i, N}\|^2 \leq \frac{2}{a_{\mathbb{B}} c_{\mathbb{B}}} \sum_{i=1}^{n_0^2(k_R+1)(k_L+1)} \|\Gamma_{N, \mathbb{B}}^{(R)}(X_{i, N})\|^2 = \frac{2}{a_{\mathbb{B}} c_{\mathbb{B}}}, \quad (47)$$

by Lemma 2.16. Hence, by the compactness, there is a subsequence  $\{N_m\}_m$  such that  $\lim_{m \rightarrow \infty} X_{i, N_m} = X_{i, \infty} \in \hat{P}_R^{(n_0, k_R, k_L)}(M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_L^{(n_0, k_R, k_L)}$ , for all  $i = 1, \dots, n_0^2(k_R + 1)(k_L + 1)$ .

By Lemma 3.14, we have

$$\begin{aligned} \omega(A) &= \lim_{m \rightarrow \infty} \text{Tr} D_{N_m} A = \lim_{m \rightarrow \infty} \text{Tr}(\tau_{N_m}(D_{N_m}) \tau_{N_m}(A)) = \lim_{m \rightarrow \infty} \sum_i \left\langle \Gamma_{N_m, \mathbb{B}}^{(R)}(X_{i, N_m}), \tau_{N_m}(A) \Gamma_{N_m, \mathbb{B}}^{(R)}(X_{i, N_m}) \right\rangle \\ &= \sum_i \text{Tr}(e_{\mathbb{B}}(X_{i, \infty} \mathbb{L}_{\mathbb{B}}(A) X_{i, \infty}^*)), \quad A \in \mathcal{A}_{(-\infty, -1]}^{\text{loc}}. \end{aligned}$$

From this, we have

$$\sum_i \text{Tr}\left(e_{\mathbb{B}}\left(X_{i, \infty} \rho_{\mathbb{B}}^{\frac{1}{2}}(1) \rho_{\mathbb{B}}^{\frac{1}{2}} X_{i, \infty}^*\right)\right) = 1,$$

and

$$\sigma_L(\cdot) := \sum_i \text{Tr} \left( e_{\mathbb{B}} \left( X_{i,\infty} \rho_{\mathbb{B}}^{\frac{1}{2}}(\cdot) \rho_{\mathbb{B}}^{\frac{1}{2}} X_{i,\infty}^* \right) \right).$$

defines a state  $\sigma_L$  on  $M_{n_0(k_L+1)} \simeq M_{n_0} \otimes \hat{P}_L^{(n_0, k_R, k_L)} M_{k_R+k_L+1} \hat{P}_L^{(n_0, k_R, k_L)}$ . We have  $\Xi_L(\sigma_L)(A) = \omega$ .  $\square$

Similarly, we have the following.

**Lemma 3.17.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Then for any  $m \geq m_{\mathbb{B}}$ ,  $\mathcal{S}(H_{\Phi_m, \mathbb{B}})$  consists of a unique state  $\omega_{\mathbb{B}, \infty}$  such that*

$$\omega_{\mathbb{B}, \infty}(A) = \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\} \times l} \left\langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \right\rangle \varphi_{\mathbb{B}} \left( \widehat{B_{\mu^{(l)}}} e_{\mathbb{B}} \left( \widehat{B_{\nu^{(l)}}} \right)^* \right), A \in \mathcal{A}_{[i, i+l-1]}, i \in \mathbb{Z}, l \in \mathbb{N}.$$

The latter property immediately means that  $\omega_{\mathbb{B}, \infty}$  is translation invariant.

*Remark 3.18.* The last representation can be written

$$\omega_{\mathbb{B}, \infty}(A) = \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\} \times l} \left\langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \right\rangle \varphi_{\mathbb{B}} \left( \left( \widehat{\omega_{\mu^{(l)}}} \otimes E_{00}^{(k_R, k_L)} \right) e_{\mathbb{B}} \left( \widehat{\omega_{\nu^{(l)}}} \otimes E_{00}^{(k_R, k_L)} \right)^* \right), A \in \mathcal{A}_{[i, i+l-1]}, i \in \mathbb{Z}, l \in \mathbb{N}.$$

Next we study how the information of support of  $\sigma_L \in \mathcal{E}_{n_0(k_L+1)}$  is reflected to state  $\Xi_L(\sigma_L)$ .

**Lemma 3.19.** *If  $\sigma_L \in \mathcal{E}_{n_0(k_L+1)}$  (resp.  $\sigma_R \in \mathcal{E}_{n_0(k_R+1)}$ ) is faithful, then*

$$\inf \{ \sigma(\Xi_L(\sigma_L)|_{\mathcal{A}_{[-l, -1]}}) \setminus \{0\} \mid l \in \mathbb{N} \} > 0, \quad (\text{resp. } \inf \{ \sigma(\Xi_R(\sigma_R)|_{\mathcal{A}_{[0, l-1]}}) \setminus \{0\} \mid l \in \mathbb{N} \} > 0.)$$

**Proof.** Let  $\sigma_L \in \mathcal{E}_{n_0(k_L+1)}$  be a faithful state. We denote the density matrix of  $\sigma_L$  by  $\tilde{\sigma}_L$ . Let  $l_{\mathbb{B}} \leq l \in \mathbb{N}$  and  $\zeta \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n$ . By the definition, we have

$$\begin{aligned} \Xi_L(\sigma_L)(\tau_{-l}(|\zeta\rangle\langle\zeta|)) &= \sigma_L \left( y_{\mathbb{B}}^{\frac{1}{2}} L_{\mathbb{B}} (\tau_{-l}(|\zeta\rangle\langle\zeta|)) y_{\mathbb{B}}^{\frac{1}{2}} \right) = \sum_{\mu^{(l)}, \nu^{(l)}} \left\langle \widehat{\psi_{\mu^{(l)}}}, \zeta \right\rangle \left\langle \zeta, \widehat{\psi_{\nu^{(l)}}} \right\rangle \sigma_L \left( y_{\mathbb{B}}^{\frac{1}{2}} \left( \widehat{B_{\nu^{(l)}}} \right)^* \rho_{\mathbb{B}} \widehat{B_{\mu^{(l)}}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \\ &= \sum_{\alpha, \beta=1}^{n_0} \sum_{i, j=0}^{k_L} \left\langle \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)}, \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \sum_{\nu^{(l)}} \left\langle \zeta, \widehat{\psi_{\nu^{(l)}}} \right\rangle \left( \widehat{B_{\nu^{(l)}}} \right)^* \rho_{\mathbb{B}}^{\frac{1}{2}} \left( \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)} \right) \right\rangle \\ &\quad \left\langle \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)}, \rho_{\mathbb{B}}^{\frac{1}{2}} \sum_{\mu^{(l)}} \left\langle \widehat{\psi_{\mu^{(l)}}}, \zeta \right\rangle \widehat{B_{\mu^{(l)}}} y_{\mathbb{B}}^{\frac{1}{2}} \tilde{\sigma}_L^{\frac{1}{2}} \left( \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)} \right) \right\rangle \\ &= \sum_{\alpha, \beta=1}^{n_0} \sum_{i, j=0}^{k_L} \left| \left\langle \zeta, \Gamma_{l, \mathbb{B}}^{(R)} \left( \rho_{\mathbb{B}}^{\frac{1}{2}} \left( e_{\beta\alpha}^{(n_0)} \otimes E_{ji}^{(k_R, k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \right\rangle \right|^2. \end{aligned}$$

From this equality, we see  $\zeta \in \overline{\tau_l(\Xi_L(\sigma_L)|_{\mathcal{A}_{[-l, -1]}})} \bigotimes_{i=0}^{l-1} \mathbb{C}^n$  if and only if

$$\zeta \in \left( \Gamma_{l, \mathbb{B}}^{(R)} \left( \rho_{\mathbb{B}}^{\frac{1}{2}} \left( M_{n_0} \otimes P_L^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \right)^{\perp} = \left( \Gamma_{l, \mathbb{B}}^{(R)} \left( \left( M_{n_0} \otimes P_L^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)} \right) \right) \right)^{\perp}.$$

Here for the equality, we used the fact that  $\rho_{\mathbb{B}}, y_{\mathbb{B}}$  and  $\tilde{\sigma}_L$  are invertible in  $M_{n_0} \otimes P_L^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ . In other words, we have

$$\Gamma_{l, \mathbb{B}}^{(R)} \left( \left( M_{n_0} \otimes P_L^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)} \right) \right) = \tau_l \left( s(\Xi_L(\sigma_L)|_{\mathcal{A}_{[-l, -1]}}) \right) \bigotimes_{i=0}^{l-1} \mathbb{C}^n. \quad (48)$$

We claim that

$$\mathcal{W} = \{\Gamma_{l,\mathbb{B}}^{(R)} \left( \left( e_{\beta\alpha}^{(n_0)} \otimes E_{0j}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \mid \alpha, \beta = 1, \dots, n_0, j = 0, \dots, k_L\}$$

is a basis of  $\Gamma_{l,\mathbb{B}}^{(R)} \left( \left( M_{n_0} \otimes P_L^{(k_R,k_L)} M_{k_R+k_L+1} P_L^{(k_R,k_L)} \right) \right)$ . To see this, note that  $\mathcal{W}$  is linearly independent because  $M_{\mathbb{B}, \hat{P}_R^{(n_0,k_R,k_L)}, \hat{P}_L^{(n_0,k_R,k_L)}} \leq l_{\mathbb{B}} \leq l$ . Therefore,  $\text{span } \mathcal{W}$  is an  $n_0^2(k_L + 1)$  dimensional subspace of  $\Gamma_{l,\mathbb{B}}^{(R)} \left( \left( M_{n_0} \otimes P_L^{(k_R,k_L)} M_{k_R+k_L+1} P_L^{(k_R,k_L)} \right) \right)$ . On the other hand, for  $X \in M_{n_0} \otimes P_L^{(k_R,k_L)} M_{k_R+k_L+1} P_L^{(k_R,k_L)}$ ,  $X \in \ker \Gamma_{l,\mathbb{B}}^{(R)}$  if and only if

$$0 = \text{Tr} \left( X (\mathcal{K}_l(\mathbb{B}))^* \right) = \text{Tr} \left( X \left( M_{n_0} \otimes \text{span} \left\{ P_L^{(k_R,k_L)} \Lambda_{\lambda}^l (1+Y)^l, P_L^{(k_R,k_L)} \Lambda_{\lambda}^l (1+Y)^l I_L(G_b) \right\} \right)^* \right)$$

Therefore, we have  $\dim \left( \Gamma_{l,\mathbb{B}}^{(R)} \left( \left( M_{n_0} \otimes P_L^{(k_R,k_L)} M_{k_R+k_L+1} P_L^{(k_R,k_L)} \right) \right) \right) = n_0^2(k_L + 1)$ . Hence we have  $\Gamma_{l,\mathbb{B}}^{(R)} \left( \left( M_{n_0} \otimes P_L^{(k_R,k_L)} M_{k_R+k_L+1} P_L^{(k_R,k_L)} \right) \right) = \text{span } \mathcal{W}$ .

Define a bounded operator  $A_l$  on  $\mathbb{C}^{n_0} \otimes \mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+1}$  by

$$\begin{aligned} & \left\langle \chi_{\beta}^{(n_0)} \otimes \chi_{\alpha}^{(n_0)} \otimes f_j^{(0,k_L)}, A_l \left( \chi_{\beta'}^{(n_0)} \otimes \chi_{\alpha'}^{(n_0)} \otimes f_{j'}^{(0,k_L)} \right) \right\rangle \\ &:= \left\langle \Gamma_{l,\mathbb{B}}^{(R)} \left( \left( e_{\beta\alpha}^{(n_0)} \otimes E_{0j}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right), \Gamma_{l,\mathbb{B}}^{(R)} \left( \left( e_{\beta'\alpha'}^{(n_0)} \otimes E_{0j'}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \right\rangle, \\ & \alpha, \beta, \alpha', \beta' = 1, \dots, n_0, \quad j, j' = 0, \dots, k_L. \end{aligned}$$

We would like to bound this operator from below for large  $l$ .

Let  $J$  be the antilinear operator on  $\mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+1}$  given as the complex conjugation with respect to the basis  $\{\chi_{\alpha}^{(n_0)} \otimes f_i^{(0,k_L)}\}_{\alpha=1,\dots,n_0, i=0,\dots,k_L}$ . Set  $O_L := J \tilde{\sigma}_L^{\frac{1}{2}} J$ . As we have  $s(e_{\mathbb{B}}) = \hat{P}_R^{(n_0,k_R,k_L)} \geq \mathbb{I} \otimes E_{00}^{(k_R,k_L)}$ , there exists a strictly positive element  $b_{\mathbb{B}} \in M_{n_0}$  such that  $\hat{E}_{00}^{(k_R,k_L)} e_{\mathbb{B}} \hat{E}_{00}^{(k_R,k_L)} = b_{\mathbb{B}} \otimes E_{00}^{(k_R,k_L)}$ . Note that

$$\begin{aligned} & \left\langle \left( e_{\beta\alpha}^{(n_0)} \otimes E_{0j}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}}, \left( e_{\beta'\alpha'}^{(n_0)} \otimes E_{0j'}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right\rangle_{\mathbb{B}} = \sigma_L \left( e_{\alpha\alpha'}^{(n_0)} \otimes E_{jj'}^{(k_R,k_L)} \right) \left\langle \chi_{\beta}^{(n_0)}, b_{\mathbb{B}} \chi_{\beta'}^{(n_0)} \right\rangle \\ &= \left\langle \chi_{\beta}^{(n_0)} \otimes \chi_{\alpha}^{(n_0)} \otimes f_j^{(0,k_L)}, (b_{\mathbb{B}} \otimes O_L^* O_L) \left( \chi_{\beta'}^{(n_0)} \otimes \chi_{\alpha'}^{(n_0)} \otimes f_{j'}^{(0,k_L)} \right) \right\rangle. \end{aligned}$$

There exist numbers  $c_1, c_2 > 0$  such that  $c_1 \mathbb{I} \leq (b_{\mathbb{B}} \otimes O_L^* O_L) \leq c_2 \mathbb{I}$ .

By Lemma 2.16, we have

$$\begin{aligned} & \left| \left\langle \Gamma_{l,\mathbb{B}}^{(R)} \left( \left( e_{\beta\alpha}^{(n_0)} \otimes E_{0j}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right), \Gamma_{l,\mathbb{B}}^{(R)} \left( \left( e_{\beta'\alpha'}^{(n_0)} \otimes E_{0j'}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \right\rangle \right. \\ & \quad \left. - \left\langle \left( e_{\beta\alpha}^{(n_0)} \otimes E_{0j}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}}, \left( e_{\beta'\alpha'}^{(n_0)} \otimes E_{0j'}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right\rangle_{\mathbb{B}} \right| \\ & \leq E_{\mathbb{B}}(l) \left\| \left( b_{\mathbb{B}}^{\frac{1}{2}} \otimes O_L \right) \chi_{\beta}^{(n_0)} \otimes \chi_{\alpha}^{(n_0)} \otimes f_j^{(0,k_L)} \right\| \left\| \left( b_{\mathbb{B}}^{\frac{1}{2}} \otimes O_L \right) \left( \chi_{\beta'}^{(n_0)} \otimes \chi_{\alpha'}^{(n_0)} \otimes f_{j'}^{(0,k_L)} \right) \right\| \leq c_2 E_{\mathbb{B}}(l). \end{aligned}$$

Hence we have

$$(c_1 - c_2(n_0^4(k_L + 1)^2)E_{\mathbb{B}}(l)) \mathbb{I} \leq A_l.$$

Set  $2l_{\mathbb{B}} \leq l_1 \in \mathbb{N}$  so that  $c_2(n_0^4(k_L + 1)^2)E_{\mathbb{B}}(l) < \frac{c_1}{2}$  for all  $l \geq l_1$ . Then we have

$$\frac{c_1}{2} \mathbb{I} \leq A_l, \quad l_1 \leq l.$$

Applying Lemma C.1, and using (48), we obtain

$$\sum_{\alpha, \beta=1}^{n_0} \sum_{j=0}^{k_L} \left| \left\langle \Gamma_{l,\mathbb{B}}^{(R)} \left( \left( e_{\beta\alpha}^{(n_0)} \otimes E_{0j}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \right\rangle \left\langle \Gamma_{l,\mathbb{B}}^{(R)} \left( \left( e_{\beta\alpha}^{(n_0)} \otimes E_{0j}^{(k_R,k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \right\rangle \right| \geq \frac{c_1}{2} \tau_l \left( s \left( \Xi_L(\sigma_L) |_{\mathcal{A}_{[-l, -1]}} \right) \right), \quad (49)$$

for  $l \geq l_1$ .

To complete the proof, let  $l \geq l_1$ . Then for any  $\zeta \in \bigotimes_{i=0}^{l-1} \mathbb{C}^n$ , we have

$$\begin{aligned}
\Xi_L(\sigma_L)(\tau_{-l}(|\zeta\rangle\langle\zeta|)) &= \sigma_L\left(y_{\mathbb{B}}^{\frac{1}{2}}L_{\mathbb{B}}(\tau_{-l}(|\zeta\rangle\langle\zeta|))y_{\mathbb{B}}^{\frac{1}{2}}\right) \geq c_{\mathbb{B}} \sum_{\mu^{(l)}, \nu^{(l)}} \left\langle \widehat{\psi_{\mu^{(l)}}}, \zeta \right\rangle \left\langle \zeta, \widehat{\psi_{\nu^{(l)}}} \right\rangle \sigma_L\left(y_{\mathbb{B}}^{\frac{1}{2}}\left(\widehat{B_{\nu^{(l)}}}\right)^* \hat{P}_L^{(n_0, k_R, k_L)} \widehat{B_{\mu^{(l)}}} y_{\mathbb{B}}^{\frac{1}{2}}\right) \\
&= c_{\mathbb{B}} \sum_{\alpha, \beta=1}^{n_0} \sum_{i, j=0}^{k_L} \left\langle \chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)}, \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \sum_{\nu^{(l)}} \left\langle \zeta, \widehat{\psi_{\nu^{(l)}}} \right\rangle \left(\widehat{B_{\nu^{(l)}}}\right)^* \left(\chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)}\right) \right\rangle \\
&\quad \left\langle \chi_{\beta}^{(n_0)} \otimes f_j^{(k_R, k_L)}, \sum_{\mu^{(l)}} \left\langle \widehat{\psi_{\mu^{(l)}}}, \zeta \right\rangle \widehat{B_{\mu^{(l)}}} y_{\mathbb{B}}^{\frac{1}{2}} \tilde{\sigma}_L^{\frac{1}{2}} \left(\chi_{\alpha}^{(n_0)} \otimes f_i^{(k_R, k_L)}\right) \right\rangle \\
&\geq c_{\mathbb{B}} \sum_{\alpha, \beta=1}^{n_0} \sum_{i=0}^{k_L} \left| \left\langle \zeta, \Gamma_{l, \mathbb{B}}^{(R)} \left( \left( e_{\beta\alpha}^{(n_0)} \otimes E_{0i}^{(k_R, k_L)} \right) \tilde{\sigma}_L^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \right\rangle \right|^2 \\
&\geq c_{\mathbb{B}} \frac{c_1}{2} \left\langle \zeta, \tau_l \left( s \left( \Xi_L(\sigma_L)|_{\mathcal{A}_{[-l, -1]}} \right) \right) \zeta \right\rangle.
\end{aligned}$$

This proves the claim of the Lemma.  $\square$

**Lemma 3.20.** *If the supports of  $\sigma_L, \sigma'_L \in \mathcal{E}_{n_0(k_L+1)}$  (resp.  $\sigma_R, \sigma'_R \in \mathcal{E}_{n_0(k_R+1)}$ ) are not orthogonal, then*

$$\|\Xi_L(\sigma_L) - \Xi_L(\sigma'_L)\| < 2, \quad (\text{resp. } \|\Xi_R(\sigma_R) - \Xi_R(\sigma'_R)\| < 2).$$

**Proof.** Note that  $\hat{L}_{\mathbb{B}} : \mathcal{A}_{(-\infty, -1]} \rightarrow M_{n_0} \otimes P_L^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)} \simeq M_{n_0(k_L+1)}$  given by  $\hat{L}_{\mathbb{B}}(\cdot) := y_{\mathbb{B}}^{\frac{1}{2}} L_{\mathbb{B}}(\cdot) y_{\mathbb{B}}^{\frac{1}{2}}$  is a unital CP map. We have  $\Xi_L(\sigma) = \sigma \circ \hat{L}_{\mathbb{B}}$ .

Let  $\sigma_L, \sigma'_L \in \mathcal{E}_{n_0(k_L+1)}$  be states such that

$$\|\Xi_L(\sigma_L) - \Xi_L(\sigma'_L)\| = 2.$$

Then there exists a sequence of self-adjoint elements  $\{a_n\}$  in the unit ball of  $\mathcal{A}_{(-\infty, -1]}$  such that

$$|\Xi_L(\sigma_L)(a_n) - \Xi_L(\sigma'_L)(a_n)| \rightarrow 2.$$

As  $\hat{L}_{\mathbb{B}}$  is a unital CP map,  $\{\hat{L}_{\mathbb{B}}(a_n)\}_n$  is a sequence of self-adjoint operators in the unit ball of  $M_{n_0(k_L+1)}$ . As the unit ball of  $M_{n_0(k_L+1)}$  is compact, there exists a subsequence  $\{a'_n\}$  such that  $\hat{L}_{\mathbb{B}}(a'_n)$  converges to some self-adjoint element  $x$  in the unit ball of  $M_{n_0(k_L+1)}$ . For this  $x$ , we have

$$|\sigma_L(x) - \sigma'_L(x)| = \lim_n \left| \sigma_L \circ \hat{L}_{\mathbb{B}}(a'_n) - \sigma'_L \circ \hat{L}_{\mathbb{B}}(a'_n) \right| = \lim_n |\Xi_L(\sigma_L)(a'_n) - \Xi_L(\sigma'_L)(a'_n)| = 2$$

As  $-1 \leq x \leq 1$ , this means that  $s(\sigma_L)$  and  $s(\sigma'_L)$  are orthogonal.  $\square$

**Lemma 3.21.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Let  $m \geq m_{\mathbb{B}}$ . For any  $\psi \in \mathcal{S}_{[0, \infty)}(H_{\Phi_{m, \mathbb{B}}})$  (resp.  $\psi \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}})$ ), there exists an  $l_{\psi} \in \mathbb{N}$  such that  $\|\psi - \psi \circ \tau_{l_{\psi}}\| < 2$  (resp.  $\|\psi - \psi \circ \tau_{-l_{\psi}}\| < 2$ ).*

**Proof.** Let  $\psi \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}})$ . By Lemma 3.16, there exists a state  $\sigma_L$  on  $M_{n_0(k_R+k_L+1)} \simeq M_{n_0} \otimes P_L^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  such that  $\Xi_L(\sigma_L) = \psi$ . Let  $\hat{\sigma}_L$  be the density matrix of  $\sigma_L$ . For each  $N \in \mathbb{N}$ , define a state  $\kappa_N \in \mathcal{E}_{n_0(k_L+1)}$  given by the density matrix

$$\hat{\kappa}_N := \rho_{\mathbb{B}}^{\frac{1}{2}} T_{\mathbb{B}}^N \left( y_{\mathbb{B}}^{\frac{1}{2}} \hat{\sigma}_L y_{\mathbb{B}}^{\frac{1}{2}} \right) \rho_{\mathbb{B}}^{\frac{1}{2}}.$$

By a straight forward calculation, one can check

$$\psi \circ \tau_{-N} = \Xi_L(\kappa_N), \quad N \in \mathbb{N}.$$

Hence the existence of  $N \in \mathbb{N}$  such that

$$\text{Tr } \hat{\sigma}_L \hat{\kappa}_N \neq 0$$

implies the existence of  $N \in \mathbb{N}$  such that  $\|\psi - \psi \circ \tau_{-N}\| < 2$  from Lemma 3.20.

Therefore, to prove the Lemma, it suffices to show that for any nonzero  $\eta \in \mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+1}$ , there exists an  $N_\eta \in \mathbb{N}$  such that

$$\left\langle \eta, \rho_{\mathbb{B}}^{\frac{1}{2}} \mathcal{K}_{N_\eta}(\mathbb{B}) y_{\mathbb{B}}^{\frac{1}{2}} \eta \right\rangle \neq 0.$$

We prove this by contradiction. Suppose for some nonzero  $\eta \in \mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+1}$ , we have

$$\left\langle \eta, \rho_{\mathbb{B}}^{\frac{1}{2}} \mathcal{K}_N(\mathbb{B}) y_{\mathbb{B}}^{\frac{1}{2}} \eta \right\rangle = 0, \quad N \in \mathbb{N}.$$

In particular, we have

$$0 = \left\langle \eta, \rho_{\mathbb{B}}^{\frac{1}{2}} \left( \mathbb{I} \otimes (\Lambda_{\lambda}(1+Y))^N \right) y_{\mathbb{B}}^{\frac{1}{2}} \eta \right\rangle = \sum_{k=0}^{k_L+k_R+1} N C_k \left\langle \eta, \rho_{\mathbb{B}}^{\frac{1}{2}} \left( \mathbb{I} \otimes \Lambda_{\lambda}^N Y^k \right) y_{\mathbb{B}}^{\frac{1}{2}} \eta \right\rangle, \quad l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y) \leq N \in \mathbb{N}.$$

By Lemma C.7, for any  $\lambda \in \{\lambda_i\}_{i=0}^{k_L}$ , we obtain

$$\left\langle \eta, \rho_{\mathbb{B}}^{\frac{1}{2}} \left( \mathbb{I} \otimes \sum_{i:\lambda_i=\lambda} E_{ii}^{(k_R, k_L)} \right) y_{\mathbb{B}}^{\frac{1}{2}} \eta \right\rangle = 0.$$

Summing up for all distinct  $\lambda \in \{\lambda_i\}_{i=0}^{k_L}$ , we obtain

$$\langle \eta, \eta \rangle = \left\langle \eta, \rho_{\mathbb{B}}^{\frac{1}{2}} y_{\mathbb{B}}^{\frac{1}{2}} \eta \right\rangle = 0.$$

This is a contradiction.  $\square$

The boundary effect decays exponentially fast in these models.

**Lemma 3.22.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Let  $m \geq m_{\mathbb{B}}$ . Then there exist constants  $0 < s'_{\mathbb{B}} < 1$  and  $C'_{\mathbb{B}} > 0$  such that for any  $N \in \mathbb{N}$ ,  $\varphi_L \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}})$ , and  $\varphi_R \in \mathcal{S}_{[0, \infty)}(H_{\Phi_{m, \mathbb{B}}})$ ,*

$$\begin{aligned} |\varphi_L(\tau_{-N}(A)) - \omega_{\mathbb{B}, \infty}(A)| &\leq C'_{\mathbb{B}} (s'_{\mathbb{B}})^N \|A\|, \quad A \in \mathcal{A}_{[-\infty, -1]}, \\ |\varphi_R(\tau_N(A)) - \omega_{\mathbb{B}, \infty}(A)| &\leq C'_{\mathbb{B}} (s'_{\mathbb{B}})^N \|A\|, \quad A \in \mathcal{A}_{[0, \infty)}. \end{aligned}$$

**Proof.** We prove the first inequality. The second one can be proven in the same way. Let  $\varphi_L \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}})$ . By Lemma 3.16, there exists a state  $\sigma_L$  on  $M_{n_0(k_R+k_L+1)} \simeq M_{n_0} \otimes P_L^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  such that  $\Xi_L(\sigma_L) = \varphi_L$ . Let  $\hat{\sigma}_L$  be the density matrix of  $\sigma_L$ . For  $l, N \in \mathbb{N}$  and  $A \in \mathcal{A}_{[-l, -1]}$ , we have  $\tau_{-N}(A) \in \mathcal{A}_{[-N-l, -N-1]}$ . Therefore we have

$$\begin{aligned} \varphi_L(\tau_{-N}(A)) &= \Xi_L(\sigma_L)(\tau_{-N}(A)) = \sigma_L(y_{\mathbb{B}}^{\frac{1}{2}} \mathbb{L}_{\mathbb{B}}(\tau_{-N}(A)) y_{\mathbb{B}}^{\frac{1}{2}}) \\ &= \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\}^{\times l}} \sum_{\mu^{(N)}, \nu^{(N)} \in \{1, \dots, n\}^{\times N}} \left\langle \widehat{\psi_{\mu^{(l)}}} \otimes \widehat{\psi_{\mu^{(N)}}}, \tau_{-N}(A) \left( \widehat{\psi_{\nu^{(l)}}} \otimes \widehat{\psi_{\nu^{(N)}}} \right) \right\rangle \sigma_L \left( y_{\mathbb{B}}^{\frac{1}{2}} \left( \widehat{B_{\nu^{(l)}}} \widehat{B_{\nu^{(N)}}} \right)^* \rho_{\mathbb{B}} \widehat{B_{\mu^{(l)}}} \widehat{B_{\mu^{(N)}}} y_{\mathbb{B}}^{\frac{1}{2}} \right) \\ &= \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\}^{\times l}} \left\langle \widehat{\psi_{\mu^{(l)}}}, A \widehat{\psi_{\nu^{(l)}}} \right\rangle \sigma_L \left( y_{\mathbb{B}}^{\frac{1}{2}} (T_{\mathbb{B}}^*)^N \left( \left( \widehat{B_{\nu^{(l)}}} \right)^* \rho_{\mathbb{B}} \widehat{B_{\mu^{(l)}}} \right) y_{\mathbb{B}}^{\frac{1}{2}} \right) \\ &= \sigma_L(y_{\mathbb{B}}^{\frac{1}{2}} (T_{\mathbb{B}}^*)^N (\mathbb{L}_{\mathbb{B}}(A)) y_{\mathbb{B}}^{\frac{1}{2}}) = \text{Tr} \left( T_{\mathbb{B}}^N \left( y_{\mathbb{B}}^{\frac{1}{2}} \hat{\sigma}_L y_{\mathbb{B}}^{\frac{1}{2}} \right) (\mathbb{L}_{\mathbb{B}}(A)) \right). \end{aligned}$$

Note from Lemma 3.17 that  $\omega_{\mathbb{B},\infty}(A) = \text{Tr } e_{\mathbb{B}} L_{\mathbb{B}}(A) = \text{Tr} \left( P_{\{1\}}^{T_{\mathbb{B}}} \left( y_{\mathbb{B}}^{\frac{1}{2}} \hat{\sigma}_L y_{\mathbb{B}}^{\frac{1}{2}} \right) L_{\mathbb{B}}(A) \right)$ . Hence we obtain

$$|\omega_{\mathbb{B},\infty}(A) - \varphi_L(\tau_{-N}(A))| \leq \left\| T_{\mathbb{B}}^N \left( \mathbb{I} - P_{\{1\}}^{T_{\mathbb{B}}} \right) \right\| \|y_{\mathbb{B}}\| \|A\| \|\mathbb{L}_{\mathbb{B}}\|. \quad (50)$$

By the density of  $\cup_l \mathcal{A}_{[-l,-1]}$  in  $\mathcal{A}_{(-\infty,-1]}$ , this proves the claim.  $\square$

This corresponds to (vii) of Theorem 1.18. Let us use this to prove (viii).

**Lemma 3.23.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Let  $m \geq \mathbf{m}_{\mathbb{B}}$ . Then any element in  $\mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m,\mathbb{B}}})$  or  $\mathcal{S}_{[0,\infty)}(H_{\Phi_{m,\mathbb{B}}})$  is a factor state.*

**Proof.** We consider  $\varphi \in \mathcal{S}_{[0,\infty)}(H_{\Phi_{m,\mathbb{B}}})$ . By Theorem 2.6.10 of [BR], it suffices to show that for any  $\varepsilon > 0$  and  $l \in \mathbb{N}$ , there exists an  $L \in \mathbb{N}$  such that

$$|\varphi(AB) - \varphi(B)\varphi(A)| \leq \varepsilon \|A\| \|B\|, \quad A \in \mathcal{A}_{[0,l-1]}, \quad B \in \mathcal{A}_{[L,\infty)}. \quad (51)$$

Fix an arbitrary  $\varepsilon > 0$  and  $l \in \mathbb{N}$ . Choose  $l \leq L \in \mathbb{N}$  so that  $C'_{\mathbb{B}}(s'_{\mathbb{B}})^{L-l} < \frac{\varepsilon}{8}$  and fix. (Here,  $C'_{\mathbb{B}}$  and  $s'_{\mathbb{B}}$  are given in the previous Lemma.)

We claim

$$|\varphi(AB) - \varphi(B)\varphi(A)| \leq \frac{\varepsilon}{4} \|A\| \|B\|, \quad A \in \mathcal{A}_{[0,l-1],+}, \quad B \in \mathcal{A}_{[L,\infty)}. \quad (52)$$

If  $\varphi(A) = 0$ , then by the Cauchy-Schwartz inequality, the left hand side is 0 and the inequality holds. If  $\varphi(A) \neq 0$ , then

$$\mathcal{A}_{[0,\infty)} \ni B \mapsto \frac{\varphi(A\tau_l(B))}{\varphi(A)}$$

defines a state on  $\mathcal{A}_{[0,\infty)}$ . By the Cauchy-Schwartz inequality and Lemma 3.11, we can check that the state belongs to  $\mathcal{S}_{[0,\infty)}(H_{\Phi_{m,\mathbb{B}}})$ . We apply the previous Lemma to this state and obtain

$$\left| \frac{\varphi(AB)}{\varphi(A)} - \omega_{\mathbb{B},\infty}(B) \right| \leq C'_{\mathbb{B}}(s'_{\mathbb{B}})^{L-l} \|B\|, \quad B \in \mathcal{A}_{[L,\infty)}.$$

Considering  $A = 1$  case, we obtain

$$|\varphi(B) - \omega_{\mathbb{B},\infty}(B)| \leq C'_{\mathbb{B}}(s'_{\mathbb{B}})^{L-l} \|B\|, \quad B \in \mathcal{A}_{[L,\infty)}.$$

From these inequalities, we obtain the claim. That (51) follows from the claim is trivial.  $\square$

**Lemma 3.24.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Let  $m \geq \mathbf{m}_{\mathbb{B}}$ . There exist  $0 < C''_{\mathbb{B}}$ ,  $N_{\mathbb{B}} \in \mathbb{N}$ ,  $\omega_{R,\mathbb{B}} \in \mathcal{S}_{[0,\infty)}(H_{\Phi_{m,\mathbb{B}}})$ , and  $\omega_{L,\mathbb{B}} \in \mathcal{S}_{(-\infty,-1]}(H_{\Phi_{m,\mathbb{B}}})$ , such that*

$$\begin{aligned} \left| \frac{\text{Tr}_{[0,N-1]}(G_{N,\mathbb{B}}A)}{\text{Tr}_{[0,N-1]}(G_{N,\mathbb{B}})} - \omega_{R,\mathbb{B}}(A) \right| &\leq C''_{\mathbb{B}} s_{\mathbb{B}}^{N-l} \|A\|, \\ \left| \frac{\text{Tr}_{[0,N-1]}(G_{N,\mathbb{B}}\tau_{N-l}(A))}{\text{Tr}_{[0,N-1]}(G_{N,\mathbb{B}})} - \omega_{L,\mathbb{B}} \circ \tau_{-l}(A) \right| &\leq C''_{\mathbb{B}} s_{\mathbb{B}}^{N-l} \|A\|, \end{aligned} \quad (53)$$

for all  $l \in \mathbb{N}$ ,  $A \in \mathcal{A}_{[0,l-1]}$ , and  $N \geq \max\{l, N_{\mathbb{B}}\}$ , and

$$\begin{aligned} \inf \{ \sigma(\omega_{R,\mathbb{B}}|_{\mathcal{A}_{[0,l-1]}}) \setminus \{0\} \mid l \in \mathbb{N} \} &> 0, \\ \inf \{ \sigma(\omega_{L,\mathbb{B}}|_{\mathcal{A}_{[-l,-1]}}) \setminus \{0\} \mid l \in \mathbb{N} \} &> 0. \end{aligned} \quad (54)$$



**Proof.** Set

$$N_{\mathbb{B}} := \max\{l_{\mathbb{B}}(n, n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y), L_{\mathbb{B}}\}.$$

Fix a basis  $\{X_i\}_{i=1}^{n_0^2(k_L+1)(k_R+1)}$  of  $M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  such that

$$\langle X_i, X_j \rangle_{\mathbb{B}} = \delta_{ij}. \quad (55)$$

Then by Lemma 2.16, we have

$$\left| \langle \Gamma_{N, \mathbb{B}}^{(R)}(X_i), \Gamma_{N, \mathbb{B}}^{(R)}(X_j) \rangle - \delta_{ij} \right| \leq E_{\mathbb{B}}(N), \quad N \in \mathbb{N}. \quad (56)$$

We claim

$$\left\| G_{N, \mathbb{B}} - \sum_i \left| \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \right\rangle \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \right| \right\| \leq \left( 4 + \sqrt{3} n_0^2 (k_L + 1)(k_R + 1) \right) E_{\mathbb{B}}(N), \quad N \geq N_{\mathbb{B}}. \quad (57)$$

To see this, let  $N \geq N_{\mathbb{B}}$  and  $\xi \in \bigotimes_{j=0}^{N-1} \mathbb{C}^n$ , and  $\xi = \eta_1 + \eta_2$ ,  $\eta_1 \in \text{Ran } \Gamma_{N, \mathbb{B}}^{(R)}$ ,  $\eta_2 \in \left( \text{Ran } \Gamma_{N, \mathbb{B}}^{(R)} \right)^\perp$  its orthogonal decomposition. Then, as  $l_{\mathbb{B}}(n, n_0, k_R, k_L, \boldsymbol{\lambda}, \mathbb{D}, \mathbb{G}, Y) \leq N_{\mathbb{B}} \leq N$ , there exists an  $X \in M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  such that  $\eta_1 = \Gamma_{N, \mathbb{B}}^{(R)}(X)$  by Proposition 3.1. Using Lemma 2.16, and  $\langle X, X \rangle_{\mathbb{B}} = \sum_i |\langle X, X_i \rangle_{\mathbb{B}}|^2$ , we obtain

$$\begin{aligned} & \left| \left\langle \xi, \left( G_{N, \mathbb{B}} - \sum_i \left| \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \right\rangle \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \right| \right) \xi \right\rangle \right| = \left| \langle \Gamma_{N, \mathbb{B}}^{(R)}(X), \Gamma_{N, \mathbb{B}}^{(R)}(X) \rangle - \sum_i \left| \langle \Gamma_{N, \mathbb{B}}^{(R)}(X), \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \rangle \right|^2 \right| \\ & \leq \sqrt{2} E_{\mathbb{B}}(N) \left( \sqrt{2} \left\| \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\|^2 + \sum_i \left( \left| \langle \Gamma_{N, \mathbb{B}}^{(R)}(X), \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \rangle \right| \left\| \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\| + |\langle X, X_i \rangle_{\mathbb{B}}| \left\| \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\| \right) \right) \\ & + \left| \langle X, X \rangle_{\mathbb{B}} - \sum_i |\langle X, X_i \rangle_{\mathbb{B}}|^2 \right| \\ & \leq \sqrt{2} E_{\mathbb{B}}(N) \left( \sqrt{2} \left\| \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\|^2 + \sum_i \left( \left| \langle \Gamma_{N, \mathbb{B}}^{(R)}(X), \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \rangle \right| \left\| \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\| + \sqrt{2} \left\| \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\|^2 \right) \right) \\ & \leq \sqrt{2} E_{\mathbb{B}}(N) \left\| \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\|^2 \left( 2\sqrt{2} + \sum_i \left( \left\| \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \right\| \right) \right) \\ & \leq \sqrt{2} E_{\mathbb{B}}(N) \left\| \Gamma_{N, \mathbb{B}}^{(R)}(X) \right\|^2 \left( 2\sqrt{2} + \sqrt{\frac{3}{2}} n_0^2 (k_L + 1)(k_R + 1) \right) \\ & \leq \left( 4 + \sqrt{3} n_0^2 (k_L + 1)(k_R + 1) \right) E_{\mathbb{B}}(N) \|\xi\|^2 \end{aligned}$$

This proves the claim. We set

$$\omega_{L, \mathbb{B}} := \frac{1}{n_0^2 (k_L + 1)(k_R + 1)} \sum_{i=1}^{n_0^2 (k_L + 1)(k_R + 1)} \Xi_L(\sigma_{L, X_i}) \in \mathcal{S}_{(-\infty, -1]}(H_{\Phi_{m, \mathbb{B}}}).$$

Fix any  $l \in \mathbb{N}$  and  $A \in \mathcal{A}_{[0, l-1]}$ . Then for any  $N \geq \max\{l, N_{\mathbb{B}}\}$ , we have

$$\begin{aligned}
& \left| \frac{\text{Tr}_{[0, N-1]}(G_{N, \mathbb{B}} \tau_{N-l}(A))}{\text{Tr}_{[0, N-1]}(G_{N, \mathbb{B}})} - \omega_{L, \mathbb{B}} \circ \tau_{-l}(A) \right| \\
& \leq \left| \frac{\text{Tr}_{[0, N-1]} \left( \left( G_{N, \mathbb{B}} - \sum_{i=1}^{n_0^2(k_L+1)(k_R+1)} \left| \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \right\rangle \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \right| \right) \tau_{N-l}(A) \right)}{\text{Tr}_{[0, N-1]}(G_{N, \mathbb{B}})} \right| \\
& + \left| \frac{\sum_{i=1}^{n_0^2(k_L+1)(k_R+1)} \left( \left\langle \Gamma_{N, \mathbb{B}}^{(R)}(X_i), \tau_{N-l}(A) \Gamma_{N, \mathbb{B}}^{(R)}(X_i) \right\rangle - \Xi_L(\sigma_{L, X_i})(\tau_{-l}(A)) \right)}{\text{Tr}_{[0, N-1]}(G_{N, \mathbb{B}})} \right| \\
& \leq \left( \left( 4 + \sqrt{3} n_0^2(k_L+1)(k_R+1) \right) E_{\mathbb{B}}(N) + E_{\mathbb{B}}(N-l) F_{\mathbb{B}} \right) \|A\|
\end{aligned}$$

Here we used (57), Lemma 3.14, and the fact  $\text{Tr}_{[0, N-1]}(G_{N, \mathbb{B}}) = n_0^2(k_L+1)(k_R+1)$  for  $N \geq \max\{l, N_{\mathbb{B}}\}$ .

Set  $C := s_{\mathbb{B}}(a_{\mathbb{B}} c_{\mathbb{B}})^{-1} n_0^2(k_L + k_R + 1)^2 \sup_{|z|=s_{\mathbb{B}}} \|(z - T_{\mathbb{B}})^{-1}\|$ . Then we have  $E_{\mathbb{B}}(N) \leq C s_{\mathbb{B}}^N$ . Set

$$C''_{\mathbb{B}} = C \left( \left( 4 + \sqrt{3} n_0^2(k_L+1)(k_R+1) \right) + F_{\mathbb{B}} \right).$$

Then we have

$$\left| \frac{\text{Tr}_{[0, N-1]}(G_{N, \mathbb{B}} \tau_{N-l}(A))}{\text{Tr}_{[0, N-1]}(G_{N, \mathbb{B}})} - \omega_{L, \mathbb{B}} \circ \tau_{-l}(A) \right| \leq C''_{\mathbb{B}} s_{\mathbb{B}}^{N-l} \|A\|, \quad N \geq \max\{N_{\mathbb{B}}, l\}.$$

From Lemma 3.19, to show the last statement, it suffices to show that

$$\kappa := \frac{1}{n_0^2(k_L+1)(k_R+1)} \sum_{i=1}^{n_0^2(k_L+1)(k_R+1)} \sigma_{L, X_i}$$

is faithful. For  $\xi \in \mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+1}$ , if  $\kappa(|\xi\rangle \langle \xi|) = 0$ , we have  $e^{\frac{1}{2}} X_i \rho_{\mathbb{B}}^{\frac{1}{2}} \xi = 0$ , for all  $i = 1, \dots, n_0^2(k_L+1)(k_R+1)$ . As  $\{X_i\}_{i=1}^{n_0^2(k_L+1)(k_R+1)}$  is a basis of  $M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$ , there exist coefficients  $c_i \in \mathbb{C}$  such that

$$\sum_i c_i X_i = \left| \chi_1^{(n_0)} \otimes f_0^{(k_R, k_L)} \right\rangle \langle \xi | \rho_{\mathbb{B}}^{-\frac{1}{2}}.$$

Hence we have

$$\|\xi\|^2 e^{\frac{1}{2}} \left| \chi_1^{(n_0)} \otimes f_0^{(k_R, k_L)} \right\rangle = 0,$$

and we obtain  $\xi = 0$ . Therefore,  $\kappa$  is faithful.  $\square$

**Lemma 3.25.** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ , and  $\mathbb{B} \in \text{ClassA}$  with respect to  $(n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$ . Let  $m \geq m_{\mathbb{B}}$ . There exists a constant  $C'''_{\mathbb{B}} > 0$  satisfying the following. : Let  $M \in \mathbb{N}$  and  $\varphi$  be a state on  $\mathcal{A}_{\mathbb{Z}}$ . Assume that we have  $\varphi(\tau_i(1 - G_{m, \mathbb{B}})) = 0$  for all  $i \in \mathbb{Z}$  with  $[i, i+m-1] \subset [-M, M]^c$ . Then for any  $L \in \mathbb{N}$  with  $M+1 \leq L$  and  $A \in \mathcal{A}_{[-L+1, L-1]^c}$ , we have*

$$\left| \varphi(A) - \left( \omega_{\mathbb{B}, \infty}|_{\mathcal{A}_{(-\infty, -1]}} \otimes \omega_{\mathbb{B}, \infty}|_{\mathcal{A}_{[0, \infty)}} \right) (A) \right| \leq C'''_{\mathbb{B}} s_{\mathbb{B}}^{L-M} \|A\|. \quad (58)$$

In particular,  $\omega_{\mathbb{B}, \infty}$  satisfies the exponential decay of correlation functions.

**Proof.** Define a linear map  $\tilde{V}_l : \left( \left( \bigotimes_{i=0}^{l-1} \mathbb{C}^n \right)^{\otimes 2} \otimes (\mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+k_R+1})^{\otimes 2} \right) \rightarrow (\mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+k_R+1})^{\otimes 2}$  by

$$\tilde{V}_l(\xi \otimes \eta) := \sum_{\mu^{(l)}, \nu^{(l)} \in \{1, \dots, n\}^l} \left\langle \widehat{\psi_{\mu^{(l)}}} \otimes \widehat{\psi_{\nu^{(l)}}}, \xi \right\rangle \left( \widehat{B_{\mu^{(l)}}} \otimes \widehat{B_{\nu^{(l)}}} \right) \eta, \quad \xi \in \left( \bigotimes_{i=0}^{l-1} \mathbb{C}^n \right)^{\otimes 2}, \quad \eta \in (\mathbb{C}^{n_0} \otimes \mathbb{C}^{k_L+k_R+1})^{\otimes 2}.$$

For each  $L, l \in \mathbb{N}$  and  $A \in \mathcal{A}_{[-L-l+1, -L]} \otimes \mathcal{A}_{[L, L+l-1]}$ , we define a linear map  $\tilde{\Theta}_A : (M_{n_0} \otimes M_{k_R+k_L+1})^{\otimes 2} \rightarrow (M_{n_0} \otimes M_{k_R+k_L+1})^{\otimes 2}$  by

$$\tilde{\Theta}_A(X) = \tilde{V}_l(A \otimes X) \tilde{V}_l^*, \quad X \in (M_{n_0} \otimes M_{k_R+k_L+1})^{\otimes 2}.$$

Here we identify  $\bigotimes_{i=-L-l+1}^{-L} \mathbb{C}^n$ ,  $\bigotimes_{i=L}^{L+l-1} \mathbb{C}^n$ , with  $\bigotimes_{i=0}^{l-1} \mathbb{C}^n$ . As in the proof of Lemma 3.14, we have  $\|\tilde{\Theta}_A\| \leq \|A\| F_{\mathbb{B}}^2$ . Note that

$$(\varphi_{\mathbb{B}} \otimes \varphi_{\mathbb{B}}) \circ \tilde{\Theta}_A(e_{\mathbb{B}} \otimes e_{\mathbb{B}}) = \left( \omega_{\mathbb{B}, \infty}|_{\mathcal{A}_{(-\infty, -1]}} \otimes \omega_{\mathbb{B}, \infty}|_{\mathcal{A}_{[0, \infty)}} \right) (A). \quad (59)$$

Fix a basis  $\{X_i\}_{i=1}^{n_0^2(k_L+1)(k_R+1)}$  of  $M_{n_0} \otimes P_R^{(k_R, k_L)} M_{k_R+k_L+1} P_L^{(k_R, k_L)}$  satisfying (55). Let  $M, N \in \mathbb{N}$  with  $N - M \geq \max\{m_{\mathbb{B}}, l_{\mathbb{B}}\}$ . Then  $\{\Gamma_{N-M, \mathbb{B}}^{(R)}(X_i) \otimes \Gamma_{N-M, \mathbb{B}}^{(R)}(X_j)\}_{i,j=1}^{n_0^2(k_L+1)(k_R+1)}$  is a basis of  $\left( \tau_{-N}(G_{N-M, \mathbb{B}}) \otimes_{i=-N}^{-M-1} \mathbb{C}^n \right) \otimes \left( \tau_{M+1}(G_{N-M, \mathbb{B}}) \otimes_{i=M+1}^N \mathbb{C}^n \right)$ . (Here we again identify  $\bigotimes_{i=-N}^{-M-1} \mathbb{C}^n$ ,  $\bigotimes_{i=M+1}^N \mathbb{C}^n$ , with  $\bigotimes_{i=0}^{N-M-1} \mathbb{C}^n$ .) Let  $\xi = \sum_{i,j=1}^{n_0^2(k_L+1)(k_R+1)} c_{i,j} \Gamma_{N-M, \mathbb{B}}^{(R)}(X_i) \otimes \Gamma_{N-M, \mathbb{B}}^{(R)}(X_j) \in \left( \tau_{-N}(G_{N-M, \mathbb{B}}) \otimes_{i=-N}^{-M-1} \mathbb{C}^n \right) \otimes \left( \tau_{M+1}(G_{N-M, \mathbb{B}}) \otimes_{i=M+1}^N \mathbb{C}^n \right)$ , with  $c_{i,j} \in \mathbb{C}$ . Let  $L, l \in \mathbb{N}$  with  $M+1 \leq L \leq N-l+1$  and  $A \in \mathcal{A}_{[-L-l+1, -L]} \otimes \mathcal{A}_{[L, L+l-1]}$ . By a straightforward calculation, we obtain

$$\begin{aligned} & \langle \xi, A\xi \rangle \\ &= \sum_{ij} \sum_{i'j'} \sum_{\alpha\beta\alpha'\beta'aba'b'} \bar{c}_{ij} c_{i'j'} \\ & (\text{Tr} \otimes \text{Tr}) \left( \begin{aligned} & \left( e_{\beta\alpha}^{(n_0)} \otimes E_{ba}^{(k_R, k_L)} \right) \otimes \left( e_{\beta'\alpha'}^{(n_0)} \otimes E_{b'a'}^{(k_R, k_L)} \right) \\ & \left( T_{\mathbb{B}}^{N-L-l+1} \otimes T_{\mathbb{B}}^{L-M-1} \right) \circ \tilde{\Theta}_A \circ \left( T_{\mathbb{B}}^{L-M-1} \otimes T_{\mathbb{B}}^{N-L-l+1} \right) \left( \left( X_i^* \left( e_{\alpha\beta}^{(n_0)} \otimes E_{ab}^{(k_R, k_L)} \right) X_{i'} \right) \otimes \left( X_j^* \left( e_{\alpha'\beta'}^{(n_0)} \otimes E_{a'b'}^{(k_R, k_L)} \right) X_{j'} \right) \right) \end{aligned} \right) \end{aligned}$$

Recall the bounds  $\|\tilde{\Theta}_A\| \leq \|A\| F_{\mathbb{B}}^2$  and (22). Using (55), and the routine argument, we see that there exists a constant  $\tilde{C}_{\mathbb{B}} > 0$  such that

$$\left| \langle \xi, A\xi \rangle - \sum_{ij} |c_{ij}|^2 (\varphi_{\mathbb{B}} \otimes \varphi_{\mathbb{B}}) \circ \tilde{\Theta}_A(e_{\mathbb{B}} \otimes e_{\mathbb{B}}) \right| \leq \tilde{C}_{\mathbb{B}} (s_{\mathbb{B}}^{N-L-l} + s_{\mathbb{B}}^{L-M}) \|A\| \sum_{ij} |c_{ij}|^2. \quad (60)$$

Substituting (59), we obtain

$$\left| \langle \xi, A\xi \rangle - \sum_{ij} |c_{ij}|^2 \left( \omega_{\mathbb{B}, \infty}|_{\mathcal{A}_{(-\infty, -1]}} \otimes \omega_{\mathbb{B}, \infty}|_{\mathcal{A}_{[0, \infty)}} \right) (A) \right| \leq \tilde{C}_{\mathbb{B}} (s_{\mathbb{B}}^{N-L-l} + s_{\mathbb{B}}^{L-M}) \|A\| \sum_{ij} |c_{ij}|^2.$$

This holds for any  $M, N \in \mathbb{N}$  with  $N - M \geq \max\{m_{\mathbb{B}}, l_{\mathbb{B}}\}$ ,  $\xi = \sum_{i,j=1}^{n_0^2(k_L+1)(k_R+1)} c_{i,j} \Gamma_{N-M, \mathbb{B}}^{(R)}(X_i) \otimes \Gamma_{N-M, \mathbb{B}}^{(R)}(X_j) \in \left( \tau_{-N}(G_{N-M, \mathbb{B}}) \otimes_{i=-N}^{-M-1} \mathbb{C}^n \right) \otimes \left( \tau_{M+1}(G_{N-M, \mathbb{B}}) \otimes_{i=M+1}^N \mathbb{C}^n \right)$ ,  $L, l \in \mathbb{N}$  with  $M+1 \leq L \leq N-l+1$  and  $A \in \mathcal{A}_{[-L-l+1, -L]} \otimes \mathcal{A}_{[L, L+l-1]}$ .

For each  $N \in \mathbb{N}$ , define  $L_N := \max\{M+1, \lfloor \frac{N}{2} \rfloor\}$ . Because we have  $N - L_N, L_N \rightarrow \infty$  as  $N \rightarrow \infty$ , there exists an  $\tilde{N}_{\mathbb{B}} \in \mathbb{N}$  such that  $\tilde{C}_{\mathbb{B}} \left( s_{\mathbb{B}}^{N-L_N-1} + s_{\mathbb{B}}^{L_N-M} \right) < \frac{1}{2}$  for all  $N \geq \tilde{N}_{\mathbb{B}}$ . We claim for any  $N \geq \max\{m_{\mathbb{B}} + M, l_{\mathbb{B}} + M, \tilde{N}_{\mathbb{B}}\}$ ,  $\xi = \sum_{i,j=1}^{n_0^2(k_L+1)(k_R+1)} c_{i,j} \Gamma_{N-M,\mathbb{B}}^{(R)}(X_i) \otimes \Gamma_{N-M,\mathbb{B}}^{(R)}(X_j) \in \left( \tau_{-N}(G_{N-M,\mathbb{B}}) \otimes_{i=-N}^{-M-1} \mathbb{C}^n \right) \otimes \left( \tau_{M+1}(G_{N-M,\mathbb{B}}) \otimes_{i=M+1}^N \mathbb{C}^n \right)$ , we have  $\sum_{ij} |c_{ij}|^2 \leq 2 \|\xi\|^2$ . Applying the above observation with  $L := L_N$ ,  $l = 1$ ,  $A = 1$ , we get this claim

$$\frac{1}{2} \sum_{ij} |c_{ij}|^2 \leq \left( 1 - \tilde{C}_{\mathbb{B}} \left( s_{\mathbb{B}}^{N-L_N-1} + s_{\mathbb{B}}^{L_N-M} \right) \right) \sum_{ij} |c_{ij}|^2 \leq \|\xi\|^2.$$

Substituting this to (60), for all  $N \geq \max\{m_{\mathbb{B}} + M, l_{\mathbb{B}} + M, \tilde{N}_{\mathbb{B}}\}$ ,  $M+1 \leq L \leq N-l+1$ ,  $\xi \in \left( \tau_{-N}(G_{N-M,\mathbb{B}}) \otimes_{i=-N}^{-M-1} \mathbb{C}^n \right) \otimes \left( \tau_{M+1}(G_{N-M,\mathbb{B}}) \otimes_{i=M+1}^N \mathbb{C}^n \right)$ , and  $A \in \mathcal{A}_{[-L-l+1, -L]} \otimes \mathcal{A}_{[L, L+l-1]}$ , we have

$$\left| \langle \xi, A\xi \rangle - \|\xi\|^2 \left( \omega_{\mathbb{B},\infty}|_{\mathcal{A}_{(-\infty, -1]}} \otimes \omega_{\mathbb{B},\infty}|_{\mathcal{A}_{[0,\infty)}} \right) (A) \right| \leq 4\tilde{C}_{\mathbb{B}} \left( s_{\mathbb{B}}^{N-L-l} + s_{\mathbb{B}}^{L-M} \right) \|A\| \|\xi\|^2. \quad (61)$$

Let  $M \in \mathbb{N}$  and  $\varphi$  be a state on  $\mathcal{A}_{\mathbb{Z}}$ . Assume that we have  $\varphi(\tau_i(1 - G_{m,\mathbb{B}})) = 0$  for all  $i \in \mathbb{Z}$  with  $[i, i+m-1] \subset [-M, M]^c$ . Let  $L \geq M+1$ ,  $l \in \mathbb{N}$  and  $A \in \mathcal{A}_{[-L-l+1, L+l-1]} \setminus [-L+1, L-1]$ . For any  $N \in \mathbb{N}$  with  $N \geq \max\{m_{\mathbb{B}} + M, l_{\mathbb{B}} + M, \tilde{N}_{\mathbb{B}}, L+l-1\}$ , the density matrix of  $\rho$  of the restriction of  $\varphi$  to  $\mathcal{A}_{[-N, -M-1]} \otimes \mathcal{A}_{[M+1, N]}$  can be written as  $\rho = \sum_i |\xi_i\rangle \langle \xi_i|$  with mutually orthogonal vectors  $\xi_i \in \left( \tau_{-N}(G_{N-M,\mathbb{B}}) \otimes_{i=-N}^{-M-1} \mathbb{C}^n \right) \otimes \left( \tau_{M+1}(G_{N-M,\mathbb{B}}) \otimes_{i=M+1}^N \mathbb{C}^n \right)$ . Applying (61) to  $\xi_i$ s, we obtain

$$\left| \varphi(A) - \left( \omega_{\mathbb{B},\infty}|_{\mathcal{A}_{(-\infty, -1]}} \otimes \omega_{\mathbb{B},\infty}|_{\mathcal{A}_{[0,\infty)}} \right) (A) \right| \leq 4\tilde{C}_{\mathbb{B}} \left( s_{\mathbb{B}}^{N-L-l} + s_{\mathbb{B}}^{L-M} \right) \|A\|. \quad (62)$$

Taking  $N \rightarrow \infty$  limit, we obtain the result.  $\square$

**Proof of Theorem 1.18.** That  $m_{\mathbb{B}} \leq 2l_{\mathbb{B}}(n, n_0, k_R, k_L, \lambda, \mathbb{D}, \mathbb{G}, Y)$  and (i), (ii) are in Proposition 3.1. (iii), (iv), (v), (vii), and (viii) are in Lemma 3.17, Lemma 3.24, Lemma 3.21, Lemma 3.22 and Lemma 3.23, respectively. (vi) is from Lemma 3.11 and Lemma 3.16. (ix) is Lemma 3.25.  $\square$

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## A Notations

Throughout the article  $\inf \emptyset = +\infty$ . We denote the Euclidean distance between a point  $x$  and a subset  $S$  in  $\mathbb{C}$  (resp.  $\mathbb{R}$ ) by  $d_{\mathbb{C}}(x, S)$  (resp.  $d_{\mathbb{R}}(x, S)$ ). For a subset  $S$  of  $\mathbb{C}$  and  $\delta > 0$ , the  $\delta$ -neighbourhood of  $S$  is denoted by  $S_{\delta}$ . We denote the open ball in  $\mathbb{C}$  centered at  $x \in \mathbb{C}$  with radius  $r$  by  $\mathcal{B}_r(x)$ .

For a linear space  $\mathcal{V}$ ,  $\dim \mathcal{V}$  denotes the dimension of  $\mathcal{V}$ . For a vector space  $\mathcal{V}$  and a set of its elements  $\{v_{\alpha}\}_{\alpha} \subset \mathcal{V}$ ,  $\text{span}\{v_{\alpha}\}_{\alpha}$  denotes the linear span of  $\{v_{\alpha}\}_{\alpha}$  in  $\mathcal{V}$ . We set  $\text{span} \emptyset := \{0\}$ .

For  $k \in \mathbb{N}$ , the set of all  $k \times k$  matrices over  $\mathbb{C}$  is denoted by  $M_k$ , while  $\text{UT}_k$  (resp.  $\text{DT}_k$ ) denotes the set of all upper (resp. lower) triangular  $k \times k$  matrices. Furthermore,  $\text{UT}_{0,k}$  (resp.  $\text{DT}_{0,k}$ ) denotes the set of elements in  $\text{UT}_k$  (resp.  $\text{DT}_k$ ) with 0 diagonal elements. Let  $\mathfrak{E}_k$  denote the set of states on  $M_k$ .

For  $A \in M_k$ ,  $\|A\|$  denotes the uniform norm of  $A$ , while  $\|A\|_2$  denotes the Hilbert Schmidt norm. The set of orthogonal projections in  $M_k$  is denoted by  $\mathcal{P}(M_k)$  and the set of positive elements of  $M_k$  by  $M_{k+}$ . Furthermore, we denote the set of unitary elements of  $M_k$  by  $\mathcal{U}(M_k)$ . For  $A \in M_{k+}$ ,  $s(A)$  denotes the support of  $A$ . For a positive linear functional  $\varphi$  on  $M_k$ , we denote its support by  $s(\varphi)$  as well. For  $A \in M_k$ ,  $s_l(A)$  (resp.  $s_r(A)$ ) denotes the left (resp. right) support of  $A$ . We write  $A > 0$  for  $A \in M_k$  if  $A$  is strictly positive. The rank of  $A \in M_k$  is denoted by  $\text{rank } A$ . For a projection  $p \in \mathcal{P}(M_k)$ , we denote  $1 - p$  by  $\bar{p}$ . For subsets  $S_1, S_2$  of  $M_k$ ,  $S_1 \cdot S_2$  denotes the set of matrices of the form  $A_1 A_2$  with  $A_1 \in S_1$ ,  $A_2 \in S_2$ . The symbol  $\text{Tr}$  denotes the trace of the matrix under consideration. We denote an inner product given by  $\text{Tr}$  by  $\langle \cdot, \cdot \rangle_{\text{Tr}}$ , i.e.,  $\langle A, B \rangle_{\text{Tr}} = \text{Tr } A^* B$  for  $A, B \in M_k$ . For  $A \in M_k$ , we set  $\text{Ad } A : M_k \rightarrow M_k$  by  $\text{Ad } A(B) = ABA^*$ ,  $B \in M_k$ .

For  $k \in \mathbb{N}$ , we denote the standard basis of  $\mathbb{C}^k$  by  $\{\chi_i^{(k)}\}_{i=1}^k$ . We denote the matrix unit  $|\chi_i^{(k)}\rangle\langle\chi_j^{(k)}|$  by  $e_{ij}^{(k)}$ . However, when numbers  $k_R, k_L \in \mathbb{N} \cup \{0\}$  are given explicitly, we denote the standard basis of  $\mathbb{C}^{k_R+k_L+1}$  by  $\{f_i^{(k_R, k_L)}\}_{i=-k_R}^{k_L}$ , and set  $E_{ij}^{(k_R, k_L)} := |f_i^{(k_R, k_L)}\rangle\langle f_j^{(k_R, k_L)}|$ . We also use the notation  $\eta_R^{(k_R, k_L)} := \sum_{i=-k_R}^{-1} f_i^{(k_R, k_L)}$  and  $\eta_L^{(k_R, k_L)} := \sum_{i=1}^{k_L} f_i^{(k_R, k_L)}$ . Furthermore, we define the projections  $P_R^{(k_R, k_L)} := \sum_{i=-k_R}^0 E_{ii}^{(k_R, k_L)}$ ,  $P_L^{(k_R, k_L)} := \sum_{i=0}^{k_L} E_{ii}^{(k_R, k_L)}$ . For  $n_0 \in \mathbb{N}$ , we also set  $\hat{P}_R^{(n_0, k_R, k_L)} := \mathbb{I}_{M_{n_0}} \otimes P_R^{(k_R, k_L)}$ ,  $\hat{P}_L^{(n_0, k_R, k_L)} := \mathbb{I}_{M_{n_0}} \otimes P_L^{(k_R, k_L)}$ ,  $\hat{E}_{ij}^{(k_R, k_L)} = \mathbb{I} \otimes E_{ij}^{(k_R, k_L)}$ . For  $a = -k_R, \dots, k_L$  we set  $Q_{R,a}^{(k_R, k_L)} := \sum_{i=-k_R}^a E_{ii}^{(k_R, k_L)}$ ,  $Q_{L,a}^{(k_R, k_L)} := \sum_{i=a}^{k_L} E_{ii}^{(k_R, k_L)}$ ,  $\hat{Q}_{R,a}^{(n_0, k_R, k_L)} := \mathbb{I}_{M_{n_0}} \otimes \sum_{i=-k_R}^a E_{ii}^{(k_R, k_L)}$  and  $\hat{Q}_{L,a}^{(n_0, k_R, k_L)} := \mathbb{I}_{M_{n_0}} \otimes \sum_{i=a}^{k_L} E_{ii}^{(k_R, k_L)}$ . In this terminology, we have  $\hat{P}_U^{(n_0, k_R, k_L)} = \hat{Q}_{R,0}^{(n_0, k_R, k_L)}$  and  $\hat{P}_L^{(n_0, k_R, k_L)} = \hat{Q}_{L,0}^{(n_0, k_R, k_L)}$ . For  $X \in M_{n_0} \otimes M_{k_R+k_L+1}$  and  $i, j = -k_R, \dots, k_L$ , we write the  $(i, j)$ -element of  $X$  by  $X_{ij}$ , i.e.,  $X = \sum_{-k_R \leq i, j \leq k_L} X_{ij} \otimes E_{ij}^{(k_R, k_L)}$ . We set  $\text{CN}(n_0, k_R, k_L) := \hat{P}_L^{(n_0, k_R, k_L)} (M_{n_0} \otimes M_{k_R+k_L+1}) \hat{P}_R^{(n_0, k_R, k_L)}$ . For  $k_R, k_L \in \mathbb{N} \cup \{0\}$ , we define linear maps  $I_R^{(k_R, k_L)} : M_{k_R+1} \rightarrow M_{k_L+k_R+1}$ , and  $I_L^{(k_R, k_L)} : M_{k_L+1} \rightarrow M_{k_L+k_R+1}$  by  $I_R^{(k_R, k_L)}(E_{ij}^{(k_R, 0)}) = E_{ij}^{(k_R, k_L)}$ ,  $i, j = -k_R, \dots, 0$ ,  $I_L^{(k_R, k_L)}(E_{ij}^{(0, k_L)}) = E_{ij}^{(k_R, k_L)}$ ,  $i, j = 0, \dots, k_L$ .

For a linear operator  $T$  acting on a vector space, we denote the spectrum of  $T$  by  $\sigma(T)$ , and the spectral radius of  $T$  by  $r_T$ . For an isolated subset  $S$  of  $\sigma(T)$ , we denote the spectral projection of  $T$  onto  $S$  by  $P_S^T$ . If  $T$  is a self-adjoint operator on a Hilbert space and  $S$  a subset of  $\mathbb{R}$ , then  $\text{Proj}[T \in S]$  also indicates the spectral projection of  $T$  corresponding to  $\sigma(T) \cap S$ . For a linear map  $\Gamma$ ,  $\ker \Gamma$ , and  $\text{Ran } \Gamma$  denote the kernel and the range of  $\Gamma$  respectively.

For a linear map  $T : M_k \rightarrow M_k$ ,  $A \in M_k$  is  $T$ -invariant if  $T(A) = A$ . A linear functional  $\psi$  is  $T$ -invariant if  $\psi \circ T = \psi$ .

For a finite dimensional Hilbert space, bracket  $\langle \cdot, \cdot \rangle$  denotes the inner product of the space under consideration. We denote the set of all bounded linear operators on  $\mathcal{H}$  by  $B(\mathcal{H})$ . For a subspace  $\mathfrak{K}$ ,  $\mathfrak{K}^\perp$  means the orthogonal complement of  $\mathcal{H}$ .

For an  $n$ -tuple of matrices  $\mathbf{v} = (v_1, \dots, v_n) \in M_k^{\times n}$  and  $R \in \text{GL}(k, \mathbb{C})$ , we denote the  $n$ -tuple  $(Rv_1 R^{-1}, \dots, Rv_n R^{-1}) \in M_k^{\times n}$  by  $R\mathbf{v}R^{-1}$ .

## B Proof of Lemma 2.11

1. is obtained by the repeated use of  $v_\mu p = p v_\mu p$ . From the same relation, we see that

$$\bar{p} (\widehat{v_{\mu(N)}})^* p = \sum_{m=1}^N (\bar{p} v_{\mu_N}^* \bar{p}) \cdots (\bar{p} v_{\mu_{m+1}}^* \bar{p}) (\bar{p} v_{\mu_m}^* p) (p v_{\mu_{m-1}}^* p) \cdots (p v_{\mu_1}^* p). \quad (63)$$

For  $N \in \mathbb{N}$ ,  $m \in \{1, \dots, N\}$ ,  $\eta \in \mathbb{C}^k$ , and a CONS  $\{\chi_i^{(k)}\}_{i=1}^k$  of  $\mathbb{C}^k$ , we have

$$\begin{aligned}
& \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \left\| (\bar{p}v_{\mu_N}^* \bar{p}) \cdots (\bar{p}v_{\mu_{m+1}}^* \bar{p}) (\bar{p}v_{\mu_m}^* p) (pv_{\mu_{m-1}}^* p) \cdots (pv_{\mu_1}^* p) \eta \right\|^2 \\
&= \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \sum_{i=1}^k \left| \left\langle (\bar{p}v_{\mu_{m+1}} \bar{p}) \cdots (\bar{p}v_{\mu_N} \bar{p}) \chi_i^{(k)}, (\bar{p}v_{\mu_m}^* p) (pv_{\mu_{m-1}}^* p) \cdots (pv_{\mu_1}^* p) \eta \right\rangle \right|^2 \\
&\leq \sum_{i=1}^k \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \left\| (\bar{p}v_{\mu_{m+1}} \bar{p}) \cdots (\bar{p}v_{\mu_N} \bar{p}) \chi_i^{(k)} \right\|^2 \|v_{\mu_m}\|^2 \left\| (pv_{\mu_{m-1}}^* p) \cdots (pv_{\mu_1}^* p) \eta \right\|^2 \\
&= \left( \text{Tr} \left( T_{\mathbf{v}_{\bar{p}}}^{N-m} \left( \sum_{i=1}^k e_{ii}^{(k)} \right) \right) \right) \left\langle \eta, T_{\mathbf{v}_p}^{m-1}(1) \eta \right\rangle \left( \sum_{\mu=1}^n \|v_{\mu}\|^2 \right). \tag{64}
\end{aligned}$$

From (64) and (63), we obtain 2:

$$\begin{aligned}
& \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \left\| \bar{p} (\widehat{v_{\mu^{(N)}}})^* p \eta \right\|^2 \\
&\leq \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \left( \sum_{m=1}^N \left\| (\bar{p}v_{\mu_N}^* \bar{p}) \cdots (\bar{p}v_{\mu_{m+1}}^* \bar{p}) (\bar{p}v_{\mu_m}^* p) (pv_{\mu_{m-1}}^* p) \cdots (pv_{\mu_1}^* p) \eta \right\| \right)^2 \\
&= \sum_{m_1=1}^N \sum_{m_2=1}^N \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \left\| (\bar{p}v_{\mu_N}^* \bar{p}) \cdots (\bar{p}v_{\mu_{m_1+1}}^* \bar{p}) (\bar{p}v_{\mu_{m_1}}^* p) (pv_{\mu_{m_1-1}}^* p) \cdots (pv_{\mu_1}^* p) \eta \right\| \\
&\quad \left\| (\bar{p}v_{\mu_N}^* \bar{p}) \cdots (\bar{p}v_{\mu_{m_2+1}}^* \bar{p}) (\bar{p}v_{\mu_{m_2}}^* p) (pv_{\mu_{m_2-1}}^* p) \cdots (pv_{\mu_1}^* p) \eta \right\| \\
&\leq \sum_{m_1=1}^N \sum_{m_2=1}^N \left( \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \left\| (\bar{p}v_{\mu_N}^* \bar{p}) \cdots (\bar{p}v_{\mu_{m_1+1}}^* \bar{p}) (\bar{p}v_{\mu_{m_1}}^* p) (pv_{\mu_{m_1-1}}^* p) \cdots (pv_{\mu_1}^* p) \eta \right\|^2 \right)^{\frac{1}{2}} \\
&\quad \left( \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \left\| (\bar{p}v_{\mu_N}^* \bar{p}) \cdots (\bar{p}v_{\mu_{m_2+1}}^* \bar{p}) (\bar{p}v_{\mu_{m_2}}^* p) (pv_{\mu_{m_2-1}}^* p) \cdots (pv_{\mu_1}^* p) \eta \right\|^2 \right)^{\frac{1}{2}} \\
&\leq \left( \sum_{m=1}^N \left( \text{Tr} T_{\mathbf{v}_{\bar{p}}}^{N-m}(1) \right)^{\frac{1}{2}} \left\langle \eta, T_{\mathbf{v}_p}^{m-1}(1) \eta \right\rangle^{\frac{1}{2}} \right)^2 \sum_{\mu=1}^n \|v_{\mu}\|^2. \tag{65}
\end{aligned}$$

To see 3, we first bound  $\|\bar{p}T_{\mathbf{v}}^N(A)\|$  for  $A \in \mathbf{M}_k$ . By 1 and 2,

$$\begin{aligned}
|\langle \bar{p}T_{\mathbf{v}}^N(A)\xi, \eta \rangle| &= \left| \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \langle A(\widehat{v_{\mu^{(N)}}})^* \xi, \bar{p}(\widehat{v_{\mu^{(N)}}})^* \bar{p}\eta \rangle \right| \\
&\leq \sum_{\mu^{(N)} \in \{1, \dots, n\} \times N} \|A\| \left\| (\widehat{v_{\mu^{(N)}}})^* \xi \right\| \left\| \bar{p}(\widehat{v_{\mu^{(N)}}})^* \bar{p}\eta \right\| \\
&\leq \|A\| \sum_{\mu^{(N)}} \left( \left\| p(\widehat{v_{\mu^{(N)}}})^* p\xi \right\| + \left\| \bar{p}(\widehat{v_{\mu^{(N)}}})^* p\xi \right\| + \left\| \bar{p}(\widehat{v_{\mu^{(N)}}})^* \bar{p}\xi \right\| \right) \cdot \left\| \bar{p}(\widehat{v_{\mu^{(N)}}})^* \bar{p}\eta \right\| \\
&\leq \|A\| \cdot \left( \sum_{\mu^{(N)}} \left\| \bar{p}(\widehat{v_{\mu^{(N)}}})^* \bar{p}\eta \right\|^2 \right)^{\frac{1}{2}} \\
&\quad \left( \left( \sum_{\mu^{(N)}} \left\| p(\widehat{v_{\mu^{(N)}}})^* p\xi \right\|^2 \right)^{\frac{1}{2}} + \left( \sum_{\mu^{(N)}} \left\| \bar{p}(\widehat{v_{\mu^{(N)}}})^* \bar{p}\xi \right\|^2 \right)^{\frac{1}{2}} + \left( \sum_{\mu^{(N)}} \left\| \bar{p}(\widehat{v_{\mu^{(N)}}})^* p\xi \right\|^2 \right)^{\frac{1}{2}} \right) \\
&= \|A\| \cdot \left( \left\langle \xi, T_{\mathbf{v}_p}^N(1)\xi \right\rangle^{\frac{1}{2}} + \left\langle \xi, T_{\mathbf{v}_{\bar{p}}}^N(1)\xi \right\rangle^{\frac{1}{2}} + \left( \sum_{\mu^{(N)}} \left\| \bar{p}(\widehat{v_{\mu^{(N)}}})^* p\xi \right\|^2 \right)^{\frac{1}{2}} \right) \left\langle \eta, T_{\mathbf{v}_{\bar{p}}}^N(1)\eta \right\rangle^{\frac{1}{2}} \\
&\leq \|A\| \left\| T_{\mathbf{v}_{\bar{p}}}^N(1) \right\|^{\frac{1}{2}} \\
&\quad \left( \sup_{M \in \mathbb{N}} \left\| T_{\mathbf{v}_p}^M \right\|^{\frac{1}{2}} + \sup_{M \in \mathbb{N}} \left\| T_{\mathbf{v}_{\bar{p}}}^M \right\|^{\frac{1}{2}} + \left( \sum_{m=1}^N \left( \text{Tr } T_{\mathbf{v}_{\bar{p}}}^{N-m}(1) \right)^{\frac{1}{2}} \left\| T_{\mathbf{v}_p}^{m-1}(1) \right\|^{\frac{1}{2}} \right) \left( \sum_{\mu=1}^n \|v_{\mu}\|^2 \right)^{\frac{1}{2}} \right) \|\xi\| \|\eta\|
\end{aligned}$$

for any  $\xi, \eta \in \mathbb{C}^k$ . For the third inequality, we used the Cauchy-Schwarz inequality. From this, we obtain

$$\begin{aligned}
\|T_{\mathbf{v}}^N(A) - pT_{\mathbf{v}}^N(A)p\| &= \|\bar{p}T_{\mathbf{v}}^N(A) + pT_{\mathbf{v}}^N(A)\bar{p}\| \\
&\leq 2\|A\| \left\| T_{\mathbf{v}_{\bar{p}}}^N(1) \right\|^{\frac{1}{2}} \\
&\quad \left( \sup_{M \in \mathbb{N}} \left\| T_{\mathbf{v}_p}^M \right\|^{\frac{1}{2}} + \sup_{M \in \mathbb{N}} \left\| T_{\mathbf{v}_{\bar{p}}}^M \right\|^{\frac{1}{2}} + \left( \sum_{m=1}^N \left( \text{Tr } T_{\mathbf{v}_{\bar{p}}}^{N-m}(1) \right)^{\frac{1}{2}} \left\| T_{\mathbf{v}_p}^{m-1}(1) \right\|^{\frac{1}{2}} \right) \left( \sum_{\mu=1}^n \|v_{\mu}\|^2 \right)^{\frac{1}{2}} \right)
\end{aligned}$$

To see 4, note that as  $T_{\mathbf{v}}^M$  is positive, we have

$$\begin{aligned}
\|T_{\mathbf{v}}^M\| &= \|T_{\mathbf{v}}^M(1)\| \leq \|T_{\mathbf{v}}^M(p)\| + \|T_{\mathbf{v}}^M(\bar{p})\| \leq \|T_{\mathbf{v}}^M(p)\| + \|\bar{p}T_{\mathbf{v}}^M(\bar{p})\bar{p}\| + \|\bar{p}T_{\mathbf{v}}^M(\bar{p})p\| + \|pT_{\mathbf{v}}^M(\bar{p})\bar{p}\| + \|pT_{\mathbf{v}}^M(\bar{p})p\| \\
&\leq \|T_{\mathbf{v}_p}^M(p)\| + \|T_{\mathbf{v}_{\bar{p}}}^M(\bar{p})\| + 2\|pT_{\mathbf{v}}^M(\bar{p})p\|^{\frac{1}{2}} \|\bar{p}T_{\mathbf{v}}^M(\bar{p})\bar{p}\|^{\frac{1}{2}} + \|pT_{\mathbf{v}}^M(\bar{p})p\| \\
&\leq \sup_{M \in \mathbb{N}} \|T_{\mathbf{v}_p}^M\| + \sup_{M \in \mathbb{N}} \|T_{\mathbf{v}_{\bar{p}}}^M\| + 2 \left( \sum_{m=1}^M \left( \text{Tr } T_{\mathbf{v}_{\bar{p}}}^{M-m}(1) \right)^{\frac{1}{2}} \left\| T_{\mathbf{v}_p}^{m-1}(1) \right\|^{\frac{1}{2}} \right) \left( \sum_{\mu=1}^n \|v_{\mu}\|^2 \right)^{\frac{1}{2}} \|T_{\mathbf{v}_{\bar{p}}}^M(\bar{p})\|^{\frac{1}{2}} \\
&\quad + \left( \sum_{m=1}^M \left( \text{Tr } T_{\mathbf{v}_{\bar{p}}}^{M-m}(1) \right)^{\frac{1}{2}} \left\| T_{\mathbf{v}_p}^{m-1}(1) \right\|^{\frac{1}{2}} \right)^2 \sum_{\mu=1}^n \|v_{\mu}\|^2.
\end{aligned}$$

Here we used 2.

## C Matrix algebra and CP maps

In this section we collect useful facts about matrix algebras. See [W], for example.

**Lemma C.1.** *Let  $m, k \in \mathbb{N}$ , with  $m \leq k$ , and  $\{\xi_i\}_{i=1}^m$ , a set of vectors of  $\mathbb{C}^k$ . Let  $A$  be an  $m \times m$  matrix given by  $A = (\langle \xi_i, \xi_j \rangle)_{i,j=1}^m$ . Let  $X := \sum_{i=1}^m |\xi_i\rangle \langle \xi_i| \in M_k$  and  $P$  be the support projection of  $X$ . Suppose that there exists a positive constant  $c$  such that  $c\mathbb{I} \leq A$ . Then we have  $cP \leq X$ . In particular, we have  $\sigma(X) \setminus \{0\} \subset [c, \|X\|]$ .*

**Theorem C.2.** *Let  $T : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$  be a positive linear map. The following properties are equivalent:*

1. *There is no nontrivial orthogonal projection  $P$  such that  $T(PM_k(\mathbb{C})P) \subset PM_k(\mathbb{C})P$ .*
2. *For any nonzero  $A \geq 0$  and  $t > 0$ ,  $\exp(tT)(A) > 0$ .*

*Remark C.3.* A positive map satisfying the above equivalent conditions is said to be irreducible.

We say that  $\lambda$  is a non degenerate eigenvalue of  $T$  if the corresponding projection  $P_{\{\lambda\}}^T$  is one dimensional. Irreducible positive maps satisfy the following properties.

**Theorem C.4.** *Let  $T : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$  be a nonzero irreducible positive linear map. Then the spectral radius  $r_T$  of  $T$  is a strictly positive, non-degenerate eigenvalue with a strictly positive eigenvector  $h_T$ :*

$$T(h_T) = r_T h_T > 0.$$

**Theorem C.5.** *Let  $T : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C})$  be a unital completely positive map and let*

$$T(A) = \sum_{i=1}^n v_i A v_i^*$$

*be its Kraus decomposition. Let  $\mathbf{v} := (v_1, \dots, v_n)$ . Then the following properties are equivalent:*

1. *There exists  $l \in \mathbb{N}$  such that  $T^l(A) > 0$  for any nonzero  $A \geq 0$ .*
2. *There exists a unique faithful  $T$ -invariant state  $\varphi$ , and it satisfies*

$$\lim_{l \rightarrow \infty} T^l(A) = \varphi(A)1, \quad A \in M_k(\mathbb{C}).$$

3.  *$\sigma(T) \cap \{z \in \mathbb{C} : |z| \geq 1\} = \{1\}$ , 1 is a non degenerate eigenvalue of  $T$ , and there exists a faithful  $T$ -invariant state.*
4. *There exists  $m \in \mathbb{N}$  such that  $K_m(\mathbf{v}) = M_k(\mathbb{C})$ .*
5. *There exists  $m \in \mathbb{N}$  such that  $K_l(\mathbf{v}) = M_k(\mathbb{C})$ , for all  $l \geq m$ .*

**Lemma C.6.** *Let  $n, n_0 \in \mathbb{N}$  and  $\omega \in \text{Prim}(n, n_0)$ . Then  $r_{T_\omega} > 0$  and there exist constants  $c > 0$ ,  $0 < s < 1$ , a faithful positive linear functional  $\varphi$  on  $M_{n_0}$  and a strictly positive element  $e \in M_{n_0}$  such that*

$$\|r_{T_\omega}^{-N} T_\omega^N(A) - \varphi(A)e\| \leq cs^N \|A\|, \quad \text{for all } A \in M_{n_0}, \text{ and } N \in \mathbb{N}.$$

**Proof.** It is easy to check that for  $\omega \in \text{Prim}(n, n_0)$ ,  $T_\omega$  is irreducible, from Lemma C.2. Hence from Lemma C.4,  $r_{T_\omega} > 0$  is a non degenerate eigenvalue of  $T_\omega$  with strictly positive eigenvector  $h_\omega$ . Define  $\hat{T}_\omega := r_{T_\omega}^{-1} h_\omega^{-\frac{1}{2}} T_\omega \left( h_\omega^{\frac{1}{2}} \cdot h_\omega^{\frac{1}{2}} \right) h_\omega^{-\frac{1}{2}}$ . Then this  $\hat{T}_\omega$  is unital and satisfies condition 4 of

Theorem C.5. From Theorem C.5, there exists  $0 < s_\omega < 1$  such that  $\sigma(\hat{T}_\omega) \setminus \{1\} \subset \mathcal{B}_{s_\omega}(0)$ , 1 is a non degenerate eigenvalue of  $\hat{T}_\omega$ , and there exists a faithful  $\hat{T}_\omega$ -invariant state  $\varphi_\omega$ . We have

$$\|\hat{T}_\omega^N(A) - \varphi_\omega(A)1\| \leq s_\omega^N C_\omega \|A\|, \quad A \in M_{n_0}, \quad N \in \mathbb{N},$$

for some  $C_\omega > 0$ . The claim of Lemma can be checked immediately from this.  $\square$



**Lemma C.7.** For  $l, k, m \in \mathbb{N}$  with  $k \leq l$ , define

$$v_a(s) = \begin{pmatrix} {}_l C_a \cdot s^l \\ {}_{l+1} C_a \cdot s^{l+1} \\ \vdots \\ {}_{l+km-1} C_a \cdot s^{l+km-1} \end{pmatrix}, \quad w_a(s) = \begin{pmatrix} {}_l^a s^l \\ (l+1)^a s^{l+1} \\ \vdots \\ (l+km-1)^a s^{l+km-1} \end{pmatrix} \in \mathbb{C}^{km}, \quad a = 0, \dots, k-1, \quad s \in \mathbb{C}.$$

Let  $\{s_i\}_{i=1}^m$  be distinct elements in  $\mathbb{C} \setminus \{0\}$ . Then we have the followings.

1. The vectors  $\{v_a(s_i)\}_{a=0, \dots, k-1, i=1, \dots, m}$  are linearly independent. In particular, there exist vectors  $\zeta_{a,i} = (\zeta_{a,i}(j))_{j=0}^{km-1} \in \mathbb{C}^{km}$ ,  $a = 0, \dots, k-1$ ,  $i = 1, \dots, m$  such that

$$\sum_{j=0}^{km-1} {}_{l+j} C_{a'} \cdot s_{i'}^{l+j} \cdot \zeta_{ai}(j) = \delta_{aa'} \delta_{ii'}.$$

2. The vectors  $\{w_a(s_i)\}_{a=0, \dots, k-1, i=1, \dots, m}$  are linearly independent. In particular, there exist vectors  $\xi_{a,i} = (\xi_{a,i}(j))_{j=0}^{km-1} \in \mathbb{C}^{km}$ ,  $a = 0, \dots, k-1$ ,  $i = 1, \dots, m$  such that

$$\sum_{j=0}^{km-1} (l+j)^{a'} s_{i'}^{j+l} \xi_{ai}(j) = \delta_{aa'} \delta_{ii'}.$$

**Proof.** This can be checked by the use of Vandermonde determinant. □

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